



Updating PageRank

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Google

Indexing

- Must index key terms on each page
Robots crawl the web — software does indexing
- Inverted file structure (like book index: terms \longrightarrow to pages)

$Term_1 \rightarrow P_i, P_j, \dots$

$Term_2 \rightarrow P_k, P_l, \dots$

\vdots

Ranking

- Determine a “PageRank” for each page $P_i, P_j, P_k, P_l, \dots$
Query independent — Based only on link structure
- Query matching
 $Q = Term_1, Term_2, \dots$ produces $P_i, P_j, P_k, P_l, \dots$
- Return $P_i, P_j, P_k, P_l, \dots$ to user in order of PageRank



Google's PageRank Idea

(Sergey Brin & Lawrence Page 1998)

- Rankings are not query dependent
 - Depend only on link structure
 - Off-line calculations
- Your page P has some rank $r(P)$
- Adjust $r(P)$ higher or lower depending on ranks of pages that point to P
- Importance is not number of in-links or out-links
 - One link to P from Yahoo! is important
 - Many links to P from me is not
- Yahoo! points many places — value of link to P is diluted



PageRank

The Definition

$$r(P) = \sum_{P \in \mathcal{B}_P} \frac{r(P)}{|P|}$$

$\mathcal{B}_P = \{\text{all pages pointing to } P\}$

$|P| = \text{number of out links from } P$

Successive Refinement

Start with $r_0(P_i) = 1/n$ for all pages P_1, P_2, \dots, P_n

Iteratively refine rankings for each page

$$r_1(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_0(P)}{|P|}$$

$$r_2(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_1(P)}{|P|}$$

\vdots

$$r_{j+1}(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_j(P)}{|P|}$$



In Matrix Notation

After Step j

$$\pi_j^T = [r_j(P_1), r_j(P_2), \dots, r_j(P_n)]$$

$$\pi_{j+1}^T = \pi_j^T \mathbf{P} \quad \text{where} \quad p_{ij} = \begin{cases} 1/|P_i| & \text{if } i \rightarrow j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{PageRank} = \lim_{j \rightarrow \infty} \pi_j^T = \pi^T \quad (\text{provided limit exists})$$

It's A Markov Chain

$$\mathbf{P} = [p_{ij}] \text{ is a stochastic matrix} \quad (\text{row sums} = 1)$$

$$\text{Each } \pi_j^T \text{ is a probability distribution vector} \quad \left(\sum_i r_j(P_i) = 1 \right)$$

$$\pi_{j+1}^T = \pi_j^T \mathbf{P} \quad \text{is random walk on the graph defined by links}$$

$$\pi^T = \lim_{j \rightarrow \infty} \pi_j^T = \text{stationary probability distribution}$$



Random Surfer

Web Surfer Randomly Clicks On Links

(Back button not a link)

Long-run proportion of time on page P_i is π_i

Problems

Dead end page (nothing to click on)

π^T not well defined

Could get trapped into a cycle $(P_i \rightarrow P_j \rightarrow P_i)$

No convergence

Convergence

Markov chain must be irreducible and aperiodic

Bored Surfer Enters Random URL

Replace \mathbf{P} by $\tilde{\mathbf{P}} = \alpha\mathbf{P} + (1 - \alpha)\mathbf{E}$ $e_{ij} = 1/n$ $\alpha \approx .85$

Different $\mathbf{E} = \mathbf{e}\mathbf{v}^T$ and α allow customization & speedup



Computing π^T

A Big Problem

$$\text{Solve } \pi^T = \pi^T \mathbf{P}$$

(stationary distribution vector)

$$\pi^T (\mathbf{I} - \mathbf{P}) = \mathbf{0}$$

(too big for direct solves)

Google's PageRank is an eigenvector of a matrix of order 2.7 billion.

One of the reasons why Google is such an effective search engine is the PageRank™ algorithm, developed by Google's founders, Larry Page and Sergey Brin, when they were graduate students at Stanford University. PageRank is determined entirely by the link structure of the Web. It is recomputed about once a month and does not involve any of the actual content of Web pages or of any individual query. Then, for any particular query, Google finds the pages on the Web that match that query and lists those pages in the order of their PageRank.

Imagine surfing the Web, going from page to page by randomly choosing an outgoing link from one page to get to the next. This can lead to dead ends at pages with no outgoing links, or cycles around cliques of interconnected pages. So, a certain fraction of the time, simply choose a random page from anywhere on the Web. This theoretical random walk of the Web is a *Markov chain* or *Markov process*. The limiting probability that a dedicated random surfer visits any particular page is its PageRank. A page has high rank if it has links to and from other pages with high rank.

Let W be the set of Web pages that can be reached by following a chain of hyperlinks starting from a page at Google and let n be the number of pages in W . The set W actually varies with time, but in May 2002, n was about 2.7 billion. Let G be the n -by- n connectivity matrix of

BY CLEVE MOLER

It tells us that the largest eigenvalue of A is equal to one and that the corresponding eigenvector, which satisfies the equation

$$x = Ax,$$

exists and is unique to within a scaling factor. When this scaling factor is chosen so that

$$\sum_i x_i = 1$$

then x is the state vector of the Markov chain. The elements of x are Google's PageRank.

If the matrix were small enough to fit in MATLAB, one way to compute the eigenvector x would be to start with a good approximate solution, such as the PageRanks from the previous month, and simply repeat the assignment statement

$$x = Ax$$

until successive vectors agree to within specified tolerance. This is known as the power method and is about the only possible approach for very large n . I'm not sure how Google actually computes PageRank, but one step of the power method would require one pass over a database of Web pages, updating weighted reference counts generated by the hyperlinks between pages.



Computing π^T

A Big Problem

Solve $\pi^T = \pi^T \mathbf{P}$ (stationary distribution vector)

$\pi^T (\mathbf{I} - \mathbf{P}) = \mathbf{0}$ (too big for direct solves)

Start with $\pi_0^T = \mathbf{e}/n$ and iterate $\pi_{j+1}^T = \pi_j^T \mathbf{P}$ (power method)



Power Method to compute PageRank

$$\pi_0^T = \mathbf{e}^T / n$$

until convergence, do

$$\pi_{j+1}^T = \pi_j^T \bar{\mathbf{P}}$$

(dense computation)

end



Power Method to compute PageRank

$$\pi_0^T = \mathbf{e}^T / n$$

until convergence, do

X $\pi_{j+1}^T = \pi_j^T \bar{\mathbf{P}}$ (dense computation)

• $\pi_{j+1}^T = \alpha \pi_j^T \mathbf{P} + (1 - \alpha) \pi_j^T \mathbf{e} \mathbf{v}^T$ (sparse computation)

end



Power Method to compute PageRank

$$\pi_0^T = \mathbf{e}^T / n$$

until convergence, do

X $\pi_{j+1}^T = \pi_j^T \bar{\mathbf{P}}$ (dense computation)

X $\pi_{j+1}^T = \alpha \pi_j^T \mathbf{P} + (1 - \alpha) \pi_j^T \mathbf{e} \mathbf{v}^T$ (sparse computation)

• $\pi_{j+1}^T = \alpha \pi_j^T \mathbf{P} + (1 - \alpha) \mathbf{v}^T$ (even less computation)

end



Convergence

Can prove $\lambda_2(\bar{\mathbf{P}}) = \alpha$

(\Rightarrow asymptotic rate of convergence is rate at which $\alpha^k \rightarrow 0$)

Google

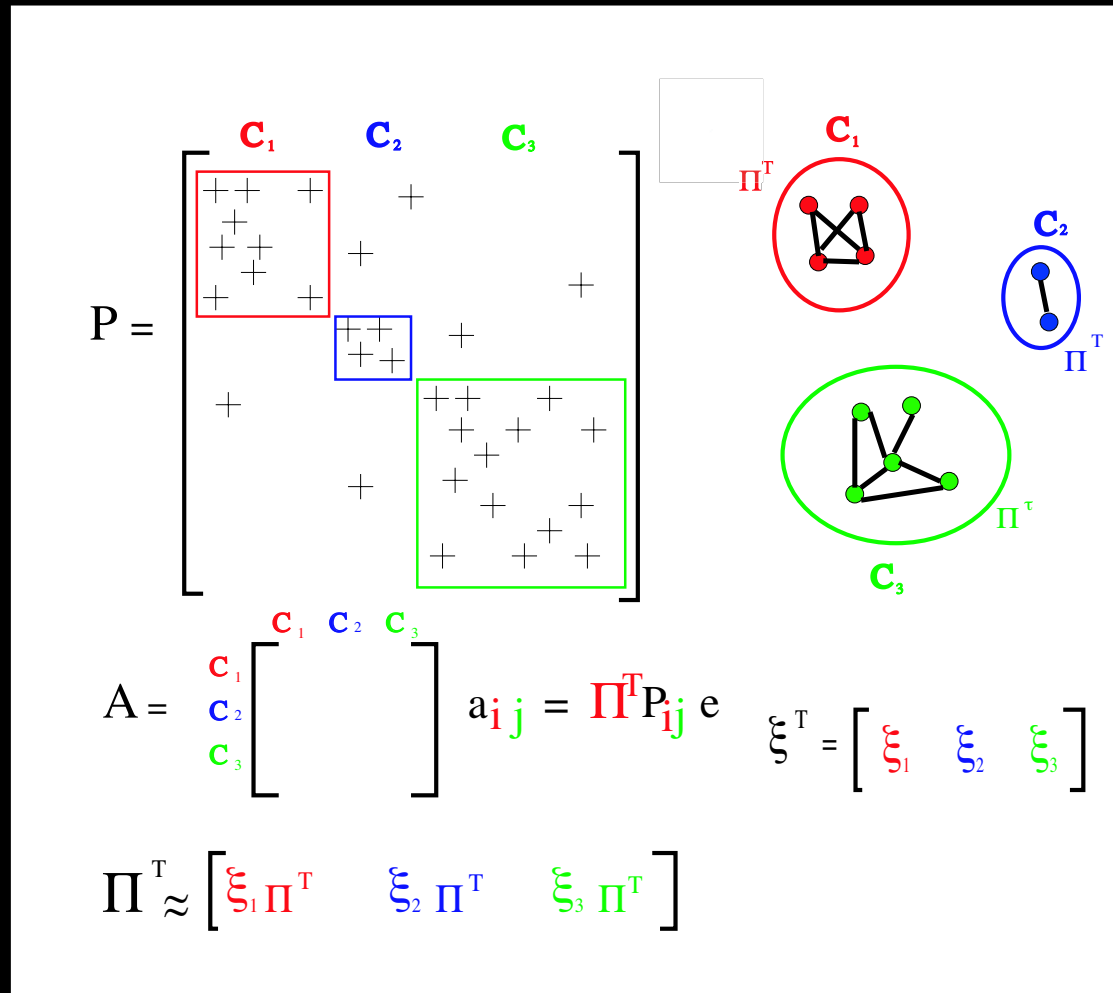
- uses $\alpha = .85$ (5/6, 1/6 interpretation)
- report 50-100 iterations til convergence
- still takes days to converge



Idea behind Aggregation

Best for NCD systems

(Simon and Ando (1960s), Courtois (1970s))



Pro

exploits structure to reduce work

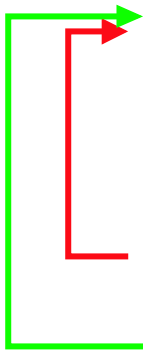
Con

produces an approximation, quality is dependent on degree of coupling



Iterative Aggregation

- Problem: repeated aggregation leads to fixed point.
- Solution: Do a power step to move off fixed point.
- Do this iteratively. Approximations improve and approach exact solution.
- Success with NCD systems, not in general.



Input: approximation to Π^T
get censored distributions $\Pi^T \Pi^T \Pi^T$
get coupling constants ξ_i

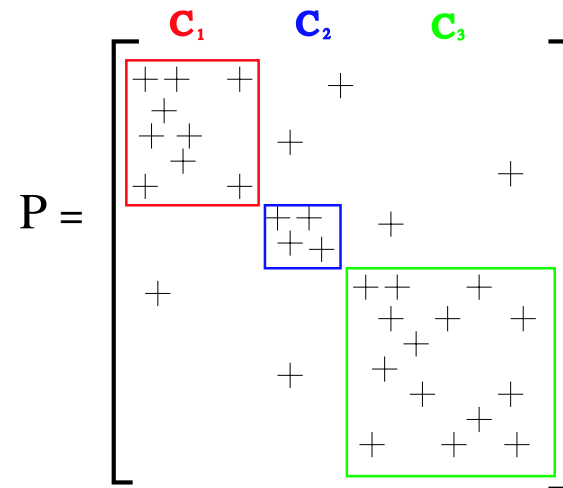
Output: get approximate global stationary distribution $\Pi^T = \left[\xi_1 \Pi^T \quad \xi_2 \Pi^T \quad \xi_3 \Pi^T \right]$

Output: move off fixed point with power step



Exact Aggregation

(Meyer 1989)



$s_i^T =$ censored (stat.) dist. of stochastic complement S_i

$$S_i = P_{ii} + P_{i*} (I - P_{**})^{-1} P_{*i}$$

For 2-level partition,

$$S_1 = P_{11} + P_{12} (I - P_{22})^{-1} P_{21}$$

$$A = \begin{matrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ c_2 & & \\ c_3 & & \end{matrix} \begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix} \quad a_{ij} = s_i^T P_{ij} e \quad \xi^T = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}$$

$$\Pi^T = \begin{bmatrix} \xi_1 s_1^T & \xi_2 s_2^T & \xi_3 s_3^T \end{bmatrix}$$

Pro

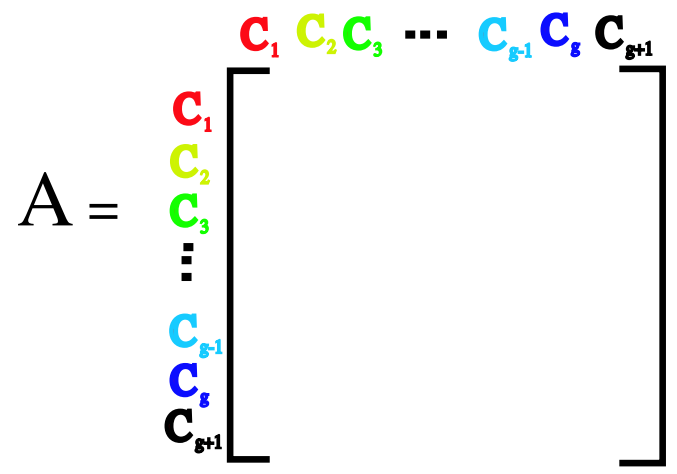
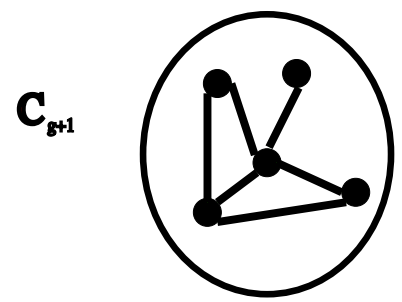
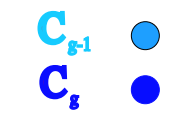
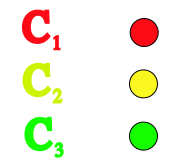
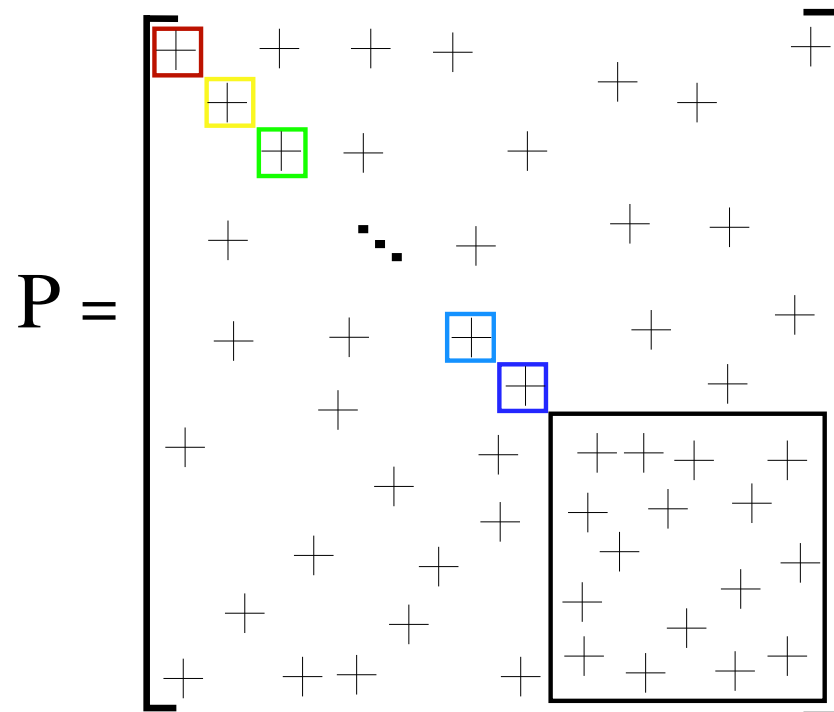
Con

only one step needed to produce exact global vector

SC matrices S_i are very expensive to compute



Back to Updating . . .





Aggregation

Partitioned Matrix

$$\mathbf{P}_{n \times n} = \begin{matrix} G & \bar{G} \\ G & \bar{G} \end{matrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & \mathbf{r}_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & \mathbf{r}_g^T \\ \mathbf{c}_1 & \cdots & \mathbf{c}_g & \mathbf{P}_{22} \end{bmatrix}$$

$$\boldsymbol{\pi}^T = (\pi_1, \dots, \pi_g \mid \pi_{g+1}, \dots, \pi_n)$$

Advantages of this Partition

$p_{11} \cdots p_{gg}$ are $1 \times 1 \implies$ Stochastic complements = 1

\implies censored distributions = 1

Only one significant complement $\mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$

Only one significant censored dist $\mathbf{s}_2^T \mathbf{S}_2 = \mathbf{s}_2^T$

A/D Theorem $\implies \mathbf{s}_2^T = (\pi_{g+1}, \dots, \pi_n) / \sum_{i=g+1}^n \pi_i$



Aggregation Matrix

$$\mathbf{A} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & \mathbf{r}_1^T \mathbf{e} \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & \mathbf{r}_g^T \mathbf{e} \\ \mathbf{s}_2^T \mathbf{c}_1 & \cdots & \mathbf{s}_2^T \mathbf{c}_g & \mathbf{s}_2^T \mathbf{P}_{22} \mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \mathbf{e} \\ \mathbf{s}_2^T \mathbf{P}_{21} & \mathbf{1} - \mathbf{s}_2^T \mathbf{P}_{21} \mathbf{e} \end{bmatrix}$$

The Aggregation/Disaggregation Theorem

If $\alpha^T = (\alpha_1, \dots, \alpha_g, \alpha_{g+1}) =$ stationary dist for \mathbf{A}

Then $\pi^T = (\alpha_1, \dots, \alpha_g \mid \alpha_{g+1} \mathbf{s}_2^T) =$ stationary dist for \mathbf{P}

Trouble! Always A Big Problem

G small $\Rightarrow \bar{G}$ big $\Rightarrow \mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{P}_{12}$ large

G big $\Rightarrow \mathbf{A}$ large



Approximate Aggregation

Assumption

Updating involves relatively few states

$$G \text{ small} \Rightarrow \mathbf{A} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}_2^T \mathbf{P}_{21} & \mathbf{1} - \mathbf{s}_2^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)} \text{ small}$$

Approximation $(\pi_{g+1}, \dots, \pi_n) \approx (\phi_{g+1}, \dots, \phi_n)$,
 where ϕ^T is old PageRank vector and π^T is new, updated PageRank

$$\mathbf{s}_2^T = \frac{(\pi_{g+1}, \dots, \pi_n)}{\sum_{i=g+1}^n \pi_i} \approx \frac{(\phi_{g+1}, \dots, \phi_n)}{\sum_{i=g+1}^n \phi_i} = \tilde{\mathbf{s}}_2^T$$

(avoids computing $\tilde{\mathbf{s}}_2^T$ for large \mathbf{S}_2)

$$\mathbf{A} \approx \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \tilde{\mathbf{s}}_2^T \mathbf{P}_{21} & \mathbf{1} - \tilde{\mathbf{s}}_2^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}$$

$$\alpha^T \approx \tilde{\alpha}^T = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1})$$

$$\pi^T \approx \tilde{\pi}^T = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_g \mid \tilde{\alpha}_{g+1} \tilde{\mathbf{s}}_2^T)$$

(not bad)



Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO

Can't do A/D twice — a fixed point emerges

Solution

Perturb A/D output to move off of fixed point

Move it in direction of solution

$$\tilde{\tilde{\pi}}^T = \tilde{\pi}^T \mathbf{P}$$

(a smoothing step)

The Iterative A/D Updating Algorithm

Determine the “ G -set” partition $\mathcal{S} = G \cup \overline{G}$

Approximate A/D step generates approximation $\tilde{\pi}^T$

Smooth the result $\tilde{\tilde{\pi}}^T = \tilde{\pi}^T \mathbf{P}$

Use $\tilde{\tilde{\pi}}^T$ as input to another approximate aggregation step

⋮



How to Partition for Updating Problem?

Intuition

- There are some bad states (G) and some good states (\overline{G}).
- Give more attention to bad states. Each state in G forms a partitioning level. Much progress toward correct PageRank is made during aggregation step.
- Lump good states in \overline{G} into 1 superstate. Progress toward correct PageRank is made during smoothing step (power iteration).



Definitions for “Good” and “Bad”

1. Good = states least likely to have π_i change
Bad = states most likely to have π_i change
2. Good = states with smallest π_i after k transient steps
Bad = states “nearby”, with largest π_i after k transient steps
3. Good = smallest π_i from old PageRank vector
Bad = largest π_i from old PageRank vector
4. Good = **fast**–converging states
Bad = **slow**–converging states



Determining “Fast” and “Slow”

Consider power method and its rate of convergence

$$\pi_{k+1}^T = \pi_k^T \mathbf{P} = \pi_k^T \mathbf{e} \pi^T + \lambda_2^k \pi_k^T \mathbf{x}_2 \mathbf{y}_2^T + \lambda_3^k \pi_k^T \mathbf{x}_3 \mathbf{y}_3^T + \cdots + \lambda_n^k \pi_k^T \mathbf{x}_n \mathbf{y}_n^T$$

Asymptotic rate of convergence is rate at which $\lambda_2^k \rightarrow 0$

Consider convergence of elements

Some states converge to stationary value faster than λ_2 -rate, due to LH e-vector \mathbf{y}_2^T .

Partitioning Rule

Put states with largest $|\mathbf{y}_2^T|_i$ values in bad group G , where they receive more individual attention in aggregation method.

Practicality

\mathbf{y}_2^T expensive, but for PageRank problem, Kamvar et al. show states with large π_i are slow-converging. \Rightarrow inexpensive soln = use old π^T to determine G .
(adaptively approximate \mathbf{y}_2^T)



Implications of Web's scale-free nature

Facts:

(1) π^T follows power law since WWW is scale-free

(experimental and theoretical justification)

(2) not all pages converge to their PageRanks at same rate

(3) pages with high PR are slow-converging

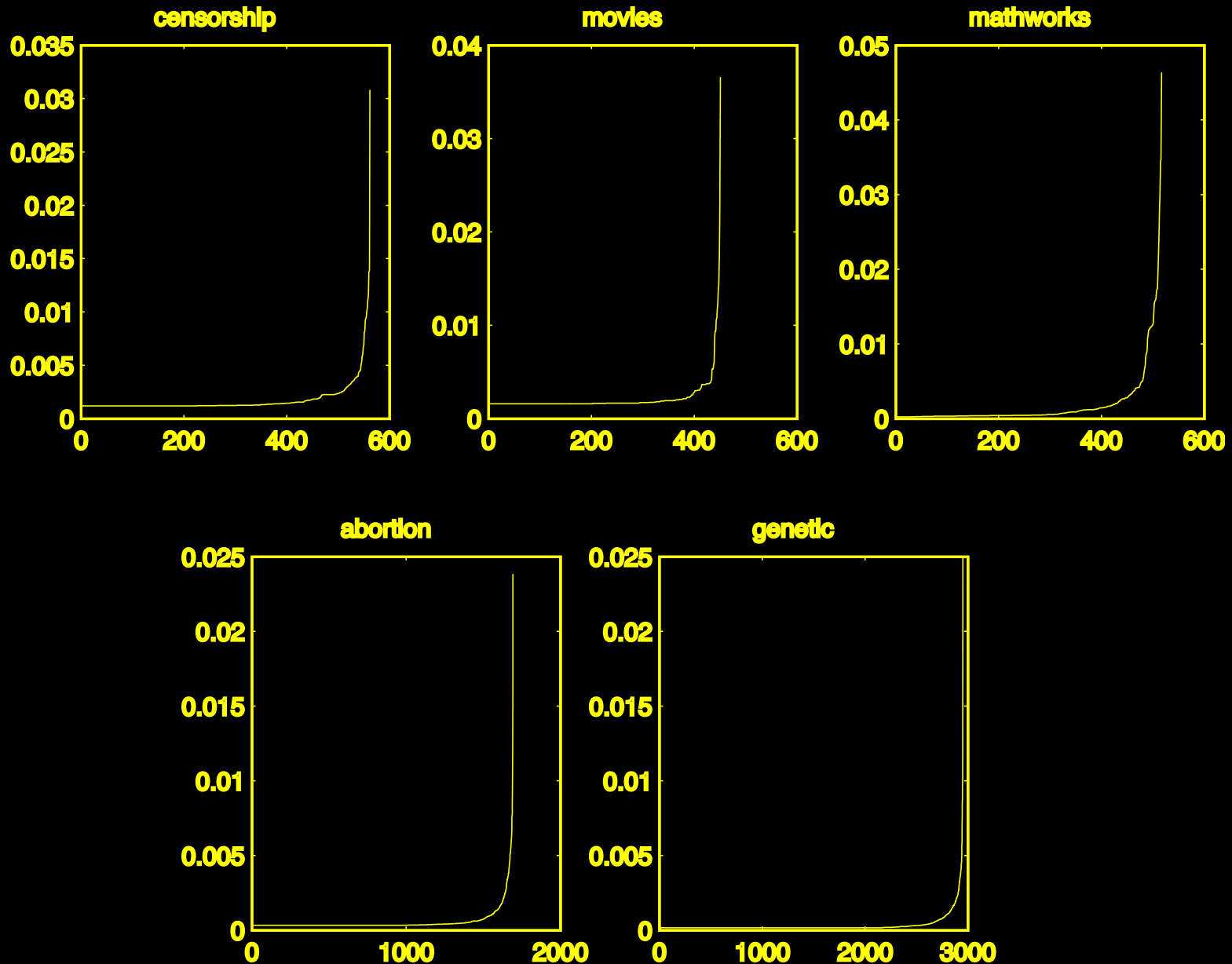
⇒ very few pages are slow-converging, but these are the pages that cause power method to drag on



Power law for PageRank

Scale-free Model of Web network creates power laws

(Kamvar, Barabasi, Raghavan)





Convergence

Theorem

Always converges to stationary dist π^T for \mathbf{P}

Converges for all partitions $\mathcal{S} = G \cup \overline{G}$

Rate of convergence is rate at which \mathbf{S}_2^n converges

$$\mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$

Dictated by Jordan structure of $\lambda_2(\mathbf{S}_2)$

$\lambda_2(\mathbf{S}_2)$ simple $\implies \pi_k^T \rightarrow \pi^T$ at the rate at which $\lambda_2^n \rightarrow 0$

The Game

Goal now is to find a relatively small G that minimizes $\lambda_2(\mathbf{S}_2)$



Convergence Findings of Ipsen/Kirkland

- ✦ For any partition, the convergence rate of IAD is at least as good as that of the power method for the Google matrix. $(\lambda_2(\mathbf{S}_2) \leq \alpha)$
- ✦ Under two rather trivial assumptions, there is always a partition so that the convergence rate of IAD is strictly smaller than that of the power method. $(\lambda_2(\mathbf{S}_2) < \alpha)$



Convergence Findings of Ipsen/Kirkland

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- ✦ Under two rather trivial assumptions, there is always a partition so that the convergence rate of IAD is strictly smaller than that of the power method. $(\lambda_2(\mathbf{S}_2) < \alpha)$

But ... how do we find partition so that $\lambda_2(\mathbf{S}_2) \ll \alpha$?



Experiments

Test Networks From Crawl Of Web

(Supplied by Ronny Lempel)

Censorship

(Sites concerning “censorship on the net”)

562 nodes 736 links

Movies

(Sites concerning “movies”)

451 nodes 713 links

MathWorks

(Supplied by Cleve Moler)

517 nodes 13,531 links

Abortion

(Sites concerning “abortion”)

1,693 nodes 4,325 links

Genetics

(Sites concerning “genetics”)

2,952 nodes 6,485 links



Parameters

Number Of Nodes (States) Added

3

Number Of Nodes (States) Removed

5

Number Of Links Added

(Different values have little effect on results)

10

Number Of Links Removed

20

Stopping Criterion

1-norm of residual $< 10^{-10}$



Censorship

Power Method

<u>Iterations</u>	<u>Time</u>
38	1.40

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	38	1.68
10	38	1.66
15	38	1.56
20	20	1.06
25	20	1.05
50	10	.69
100	8	.55
300	6	.65
400	5	.70

nodes = 562 links = 736



Censorship

Power Method

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Iterative Aggregation

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20	20	1.06
25	20	1.05
50	10	.69
100	8	.55
200	6	.53
300	6	.65
400	5	.70

nodes = 562 links = 736



Movies

Power Method

<u>Iterations</u>	<u>Time</u>
17	.40

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	12	.39
10	12	.37
15	11	.36
20	11	.35
100	9	.33
200	8	.35
300	7	.39
400	6	.47

nodes = 451 links = 713



Movies

Power Method

<u>Iterations</u>	<u>Time</u>
17	.40

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	12	.39
10	12	.37
15	11	.36
20	11	.35
25	11	.31
50	9	.31
100	9	.33
200	8	.35
300	7	.39
400	6	.47

nodes = 451 links = 713



MathWorks

Power Method

<u>Iterations</u>	<u>Time</u>
54	1.25

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	53	1.18
10	52	1.29
15	52	1.23
20	42	1.05
25	20	1.13
300	11	.83
400	10	1.01

nodes = 517 links = 13,531



MathWorks

Power Method

<u>Iterations</u>	<u>Time</u>
54	1.25

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	53	1.18
10	52	1.29
15	52	1.23
20	42	1.05
25	20	1.13
50	18	.70
100	16	.70
200	13	.70
300	11	.83
400	10	1.01

nodes = 517 links = 13,531



Abortion

Power Method

<u>Iterations</u>	<u>Time</u>
106	37.08

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

nodes = 1,693 links = 4,325



Abortion

Power Method

<u>Iterations</u>	<u>Time</u>
106	37.08

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
100	13	5.18
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

nodes = 1,693 links = 4,325



Genetics

Power Method

<u>Iterations</u>	<u>Time</u>
92	91.78

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
1000	5	17.76
1500	5	31.84

nodes = 2,952 links = 6,485



Genetics

Power Method

<u>Iterations</u>	<u>Time</u>
92	91.78

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
500	7	11.14
1000	5	17.76
1500	5	31.84

nodes = 2,952 links = 6,485



Large-Scale Implementation

Dangling Nodes *(nodes with no outlinks)*

- replacing 0 rows with \mathbf{v}^T takes too much storage.
- must be done implicitly in power method.

IAD's Aggregated System Solve

- direct vs. sparse methods

Simulating updates to Web

- how to do this accurately, and keep scale-free properties of web
- need collections of the web over time.



Conclusions

First updating algorithm to handle both element- and state-updates.

Algorithm is very sensitive to partition.

For PageRank problem, partition can be determined cheaply from old PageRanks.

For general Markov updating, use \mathbf{y}_2^T to determine partition. When too expensive, approximate adaptively with Aitken's δ^2 or difference of successive iterates.

Improvements

Practical

Optimize G -set

Accelerate Smoothing

Theoretical

Relationship between partitioning by \mathbf{y}_2^T and $\lambda_2(\mathbf{S}_2)$ not well-understood.

Predict algorithm and partitioning by old π^T will work very well on other scale-free networks.