Rank and Rating Aggregation

Amy Langville
Mathematics Department
College of Charleston

Hamilton Institute 8/6/2008

Collaborators

- Carl Meyer, N. C. State University
- College of Charleston (current and former students)
  - Luke Ingram
  - Kathryn Pedings
  - Neil Goodson
  - Colin Stephenson
  - Emmie Douglas

Outline

- Popular Ranking Methods
  - Massey
  - Colley
  - Markov
  - HITS
- New Ranking Methods
  - Rank Differential Method
  - Rating Differential Method
- Aggregation
  - Methods of Comparison
  - Rank Aggregation
  - Rating Aggregation

Popular Ranking Methods

Ranking

Problem

given data of n items, create a ranked list of these items

↖ (e.g., pairwise comparisons)

Applications

- webpages (PageRank, HITS, SALSA, ...)
- sports teams (Massey, Colley, Markov, mHITS, ...)
- recommendation systems (Netflix movies, Amazon books, ...)

Related Problem

cluster n items into groups

The Data

$$\mathbf{A} \geq 0$$

Sports Examples

$$\mathbf{A} = \text{team} \overset{\text{team}}{\begin{pmatrix} \\ \text{stats} \\ \\ \end{pmatrix}} ; \qquad \mathbf{A} = \text{team} \overset{\text{team}}{\begin{pmatrix} \text{stat} \\ \text{differentials} \\ \\ \end{pmatrix}}$$

$$\mathbf{A} = \text{game} \overset{\text{team}}{\begin{pmatrix} \text{wins} \\ \& \\ \text{losses} \\ \end{pmatrix}}$$

Popular Ranking Methods

- Massey: symmetric linear system $\mathbf{Mr} = \mathbf{p}$
- Colley: s.p.d linear system $\mathbf{Cr} = \mathbf{b}$
- Markov: irreducible eigensystem $\mathbf{Vr} = \mathbf{r}$, where $\mathbf{V} \geq 0$
- mHITS: Sinkhorn-Knopp algorithm on $\mathbf{P} \geq 0$

Popular Ranking Methods

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ranking methods are adapted to fit the application

(webpages, sports teams, movies, etc.)

Popular Ranking Methods

Markov Method

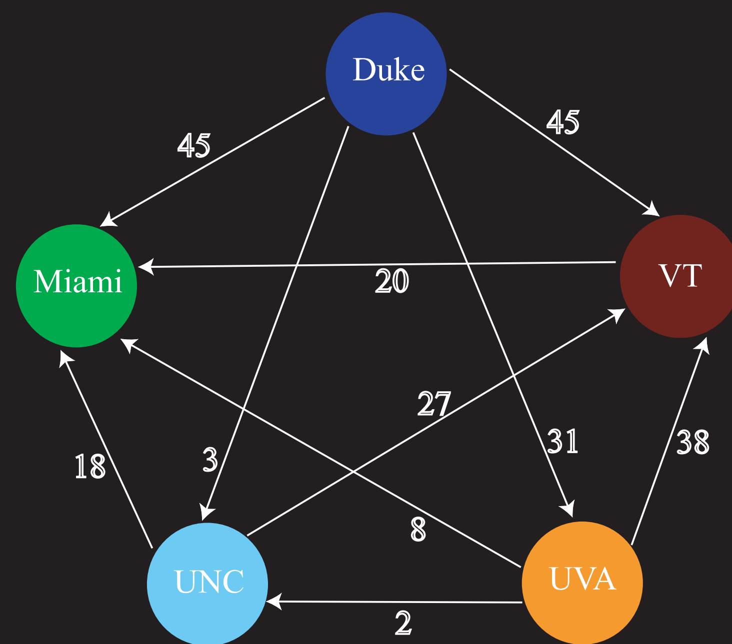
Markov Method

Voting Matrix $\mathbf{V} \geq 0$

- losers vote with points given up (or some other statistic)
- winners and losers vote with points given up
- losers vote with point differentials

$\mathbf{V} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	45	3	31	45
Miami	0	0	0	0	0
UNC	0	18	0	0	27
UVA	0	8	2	0	38
VT	0	20	0	0	0



- vote with multiple statistics

$$\mathbf{V} = \alpha_1 \mathbf{V}_1 + \alpha_2 \mathbf{V}_2 + \cdots + \alpha_k \mathbf{V}_k$$

Fair Weather Fan's Random Walk

- row normalize to make **V** stochastic

$$\mathbf{V} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 45/124 & 3/124 & 31/124 & 45/124 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 18/45 & 0 & 0 & 27/45 \\ 0 & 8/48 & 2/48 & 0 & 38/48 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Check/enforce irreducibility

- Solve eigensystem: $\mathbf{Vr} = \mathbf{r}$

Team	r	Rank
Duke	.088	5th
Miami	.442	1st
UNC	.095	4th
UVA	.110	3rd
VT	.265	2nd

- **r** = fair weather fan's long-term visit proportions

- not as successful at ranking and predicting winners as mHITS

(coming soon)

Nonnegativity and Markov

- enforce irreducibility, aperiodicity

$$\bar{\mathbf{V}} = \mathbf{V} + \epsilon\, \mathbf{e}\,\mathbf{e}^T \geq 0$$

- P-F guarantees existence and uniqueness of **r**

- P-F guarantees convergence and rate of convergence of power method on **V**

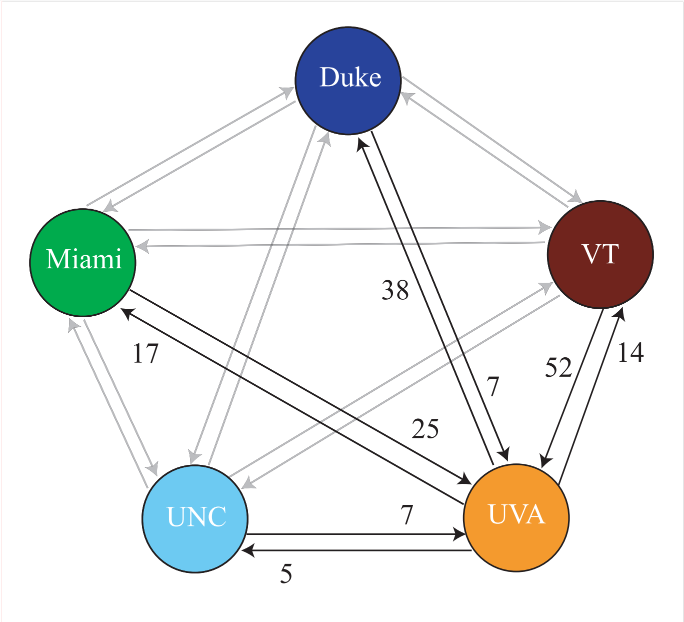
Popular Ranking Methods

mHITS Method

mHITS Method

each team i gets both offensive rating $\mathbf{o}_i$ and defensive rating $\mathbf{d}_i$

- **mHITS Thesis:** A team is a good defensive team (i.e., deserves a high defensive rating d_j) when it holds its opponents (particularly strong offensive teams) to low scores. A team is a good offensive team (i.e., deserves a high offensive rating o_i) when it scores many points against its opponents (particularly opponents with high defensive ratings).



Graph

P=

	Duke	Miami	UNC	UVA	VT
Duke	0	7	21	7	0
Miami	52	0	34	25	27
UNC	24	16	0	7	3
UVA	38	17	5	0	14
VT	45	7	30	52	0

Point Matrix $\mathbf{P} \geq 0$

mHITS Equations

Summation Notation

$$d_j = \sum_{i \in I_j} p_{ij} \frac{1}{o_i} \qquad \text{and} \qquad o_i = \sum_{j \in L_i} p_{ij} \frac{1}{d_j}$$

Matrix Notation: iterative procedure

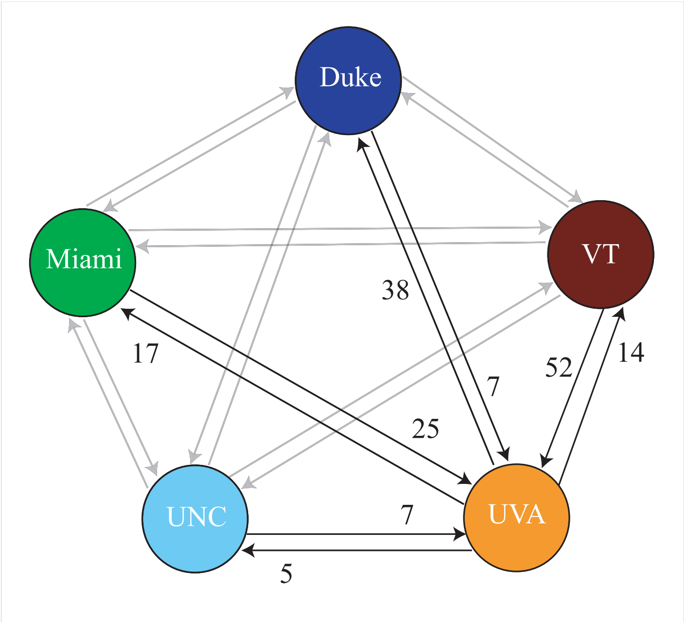
$$\mathbf{d}^{(k)} = \mathbf{P}^T \frac{1}{\mathbf{o}^{(k)}} \qquad \text{and} \qquad \mathbf{o}^{(k)} = \mathbf{P} \frac{1}{\mathbf{d}^{(k-1)}}$$

- related to the Sinkhorn-Knopp algorithm for matrix balancing
  (uses successive row and column scaling to transform $\mathbf{P} \geq 0$ into doubly stochastic matrix $\mathbf{S}$)
- P. Knight uses Sinkhorn-Knopp algorithm to rank webpages

mHITS Results: tiny NCAA

(data from Luke Ingram)

Team	Off. Rating o	Off. Rank	Def. Rating d	Def. Rank
Duke	.4	5th	140	5th
Miami	1.8	1st	67	2nd
UNC	.6	4th	97	4th
UVA	1.0	3rd	80	3rd
VT	1.4	2nd	34	1st



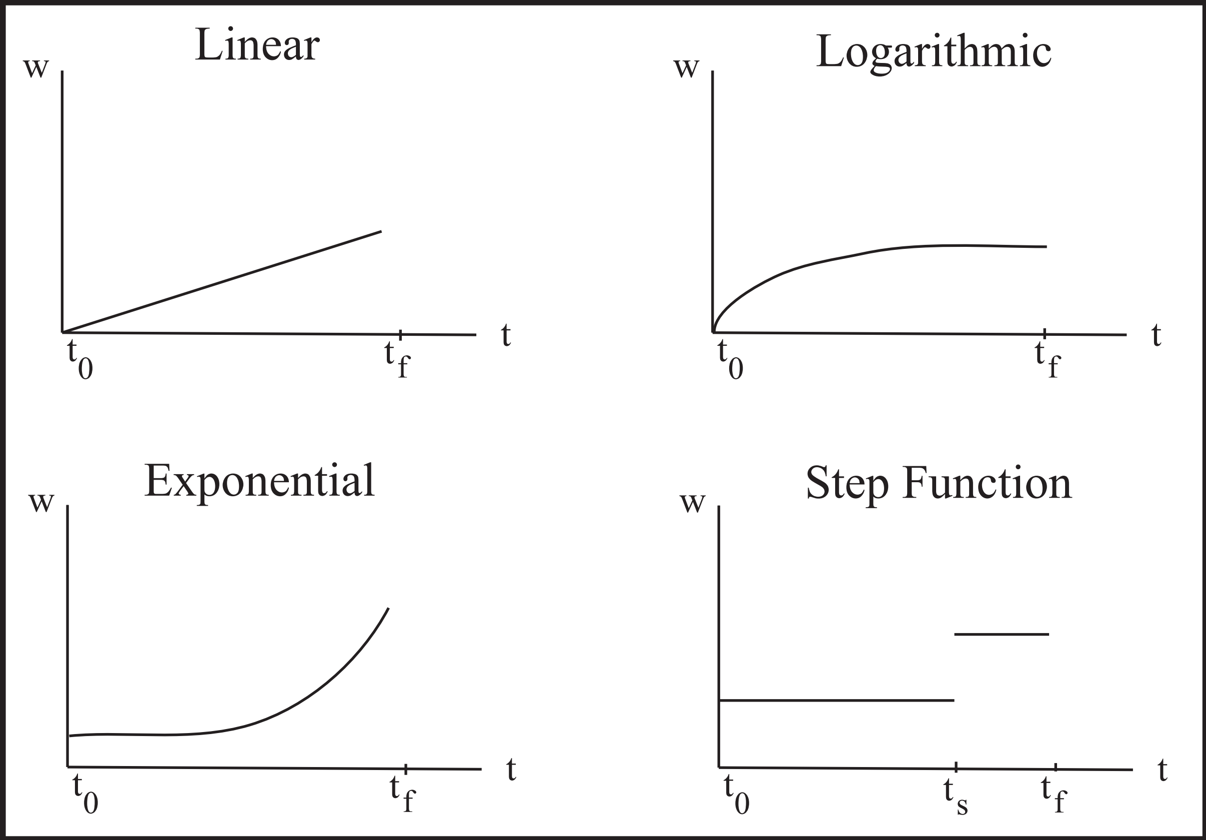
	$r = o/d$ Rating	$r = o/d$ Rank
Duke	0.003	5th
Miami	0.027	2nd
UNC	0.006	4th
UVA	0.012	3rd
VT	0.041	1st

Weighted mHITS

- Weighted Point Matrix $\bar{\mathbf{P}} \geq 0$

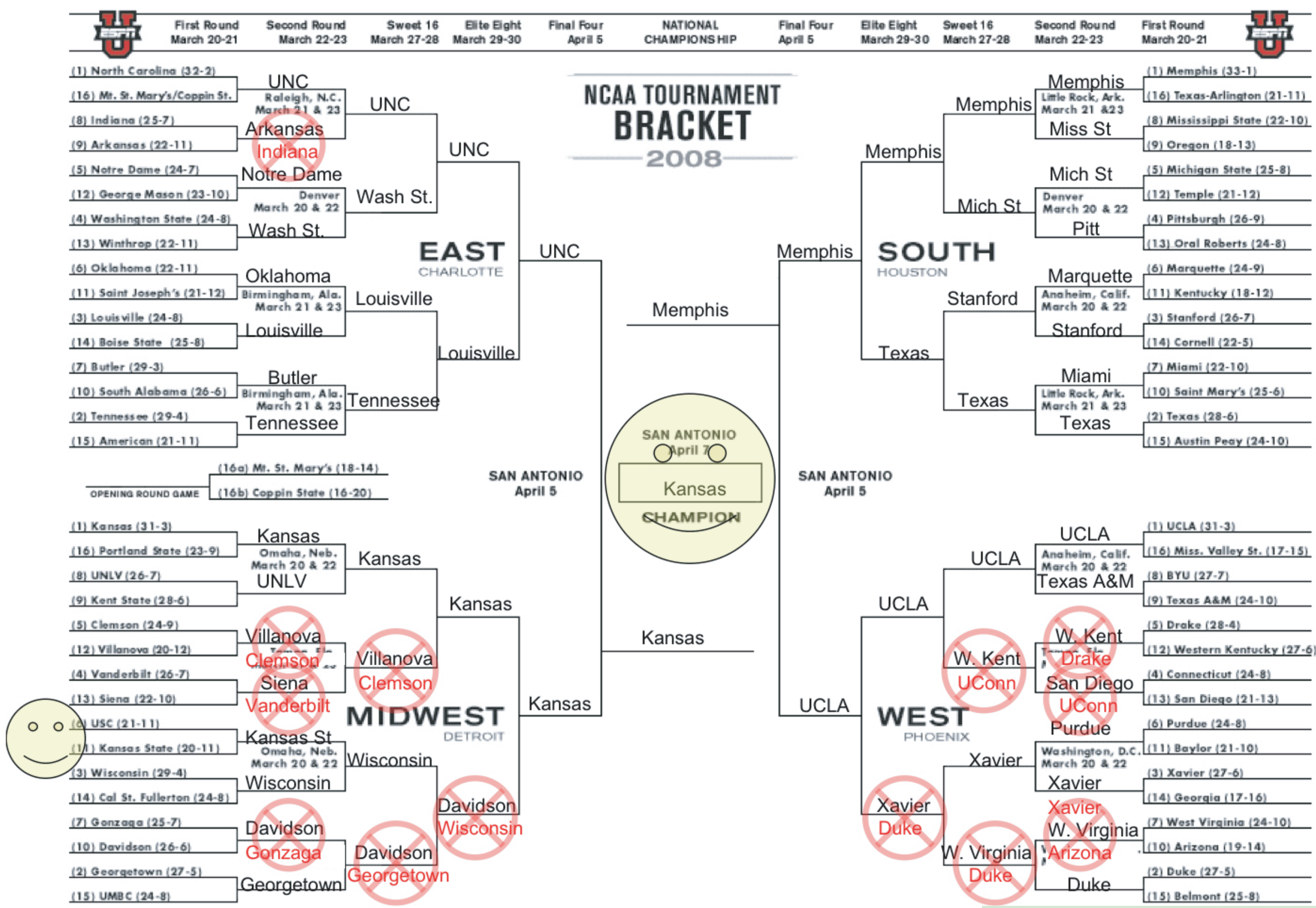
$$\bar{\mathbf{p}}_{ij} = w_{ij} \ \mathbf{p}_{ij} \qquad (w_{ij} = \text{weight of matchup between teams } i \text{ and } j)$$

- possible weightings w_{ij}



mHITS Results: full NCAA

(image from Neil Goodson and Colin Stephenson)



mHITS - Log Weighted

ESPN Pts: 1450 out of 1680
National Rank: 834 out 3.6 mil.
Percentile: 100%`

mHITS Results: full NCAA

Method	ESPN score
Massey Linear	1450
Massey Log	1450
mHITS Log	1450
Massey Step	1420
Massey Exponential	1400
mHITS Step	1320
Massey Uniform	1310
mHITS Linear	1310
mHITS Uniform	1310
Colley Linear	1100
Colley Log	1010

- weightings can easily be applied to all ranking methods

  $\Rightarrow$ interesting possibilities for weighted webpage ranking

mHITS Point Matrix $\mathbf{P} \geq \mathbf{0}$

Perron-Frobenius guarantees (for irreducible **P** with total support)

- existence of **o** and **d**
- uniqueness of **o** and **d**
- convergence of mHITS algorithm
- rate of convergence of mHITS algorithm

  $\sigma_2^2(\mathbf{S})$, where $\mathbf{S} = \mathbf{D}(1/\mathbf{o})\,\mathbf{P}\,\mathbf{D}(1/\mathbf{d})$

mHITS on Netflix

each movie i gets a rating $\mathbf{m}_i$ and each user gets a rating $\mathbf{u}_j$

- mHITS Thesis: A movie is a good (i.e., deserves a high rating m_i) if it gets high ratings from good (i.e., discriminating) users. A user is good (i.e., deserves a high rating u_j) when his or her ratings match the true rating of a movie.

mHITS Netflix Algorithm

$\mathbf{u} = \mathbf{e}$;

for i = 1 : maxiter

$\mathbf{m} = \mathbf{A}\,\mathbf{u}$;

$\mathbf{m} = \frac{5(\mathbf{m} - min(\mathbf{m}))}{max(\mathbf{m}) - min(\mathbf{m})}$;

$\mathbf{u} = \frac{1}{((\mathbf{R} - (\mathbf{R} > 0). * (\mathbf{e}\mathbf{m}^T)).^2)\mathbf{e}}$;

end

Netflix mHITS results

17770 movies, $\approx$ .5 million users

David Gleich's subset: 1500 "super users" (rate $\geq$ 1000 movies)

1^{st}	Raiders of the Lost Ark
2^{nd}	Silence of the Lambs
3^{rd}	The Sixth Sense
4^{th}	Shawshank Redemption
5^{th}	LOR: Fellowship of the Ring
6^{th}	The Matrix
7^{th}	LOR: The Two Towers
8^{th}	Pulp Fiction
9^{th}	LOR: The Return of the King
10^{th}	Forrest Gump
11^{th}	The Usual Suspects
12^{th}	American Beauty
13^{th}	Pirates of the Carribbean: Black Pearl
14^{th}	The Godfather
15^{th}	Braveheart

New Ranking Methods

Ranking Philosophies

Old

e.g.

Data

	Team 1	Team 2	. . .	Team n
Team 1	7-4	17-8		6-21
	1-5	4-2		12-10
Team 2	3-11	6-8		13-19
	1-3	15-21		9-12
.				
.				
.				
Team n	11-10	8-4		5-2
	11-9	12-8		11-8

Method

Massey
Mr = p

Rating Vector

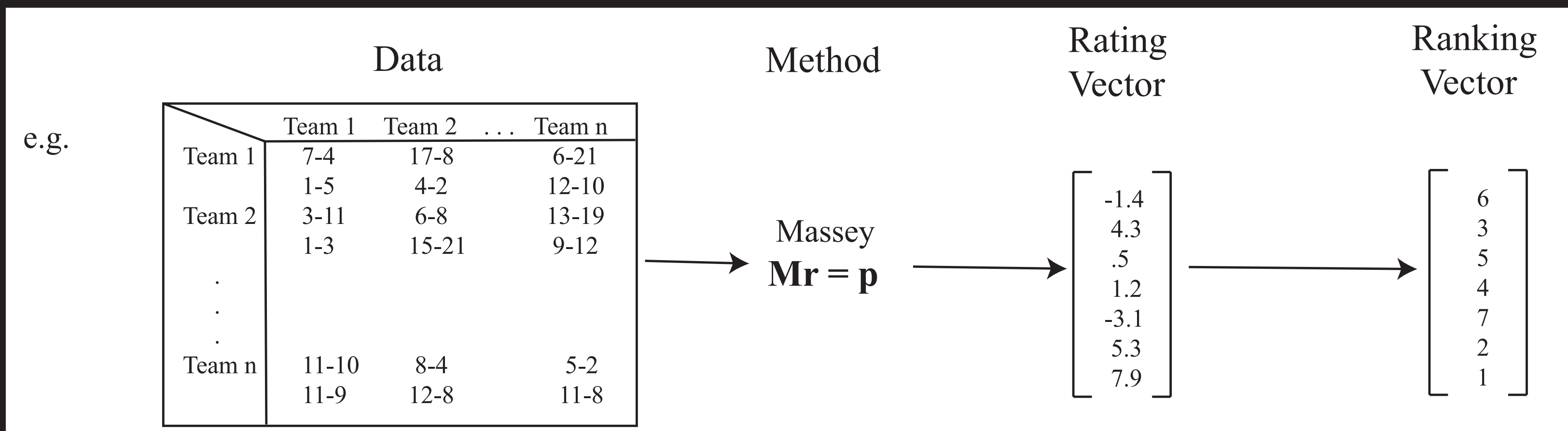
$$\begin{bmatrix} -1.4 \\ 4.3 \\ .5 \\ 1.2 \\ -3.1 \\ 5.3 \\ 7.9 \end{bmatrix}$$

Ranking Vector

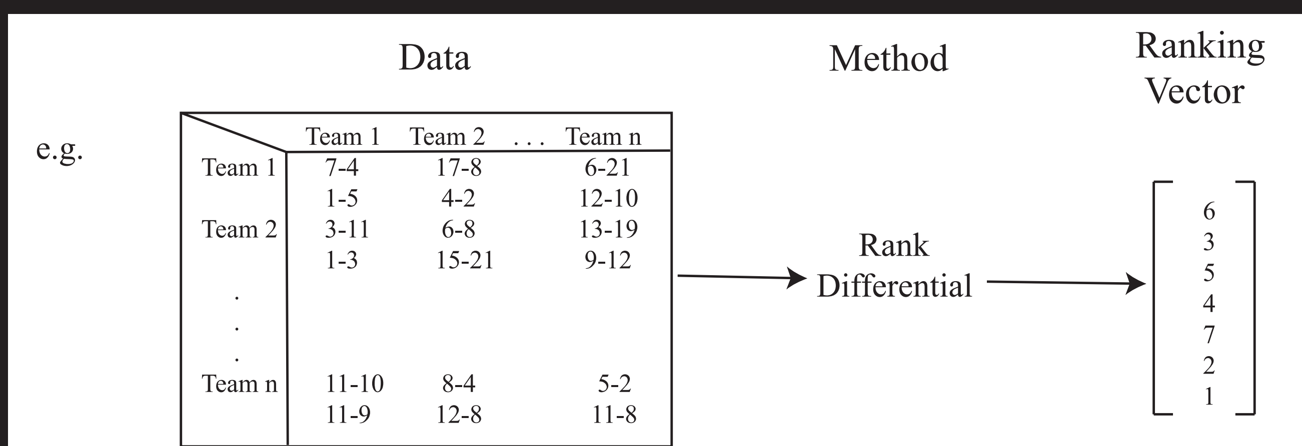
$$\begin{bmatrix} 6 \\ 3 \\ 5 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix}$$

Ranking Philosophies

Old



New



New Ranking Methods

Rank Differential Method

Ranking Vector

- Every ranking vector is a permutation
- Relative positions matter
- One-to-one mapping:

ranking vector $\left[\ \right] \longleftrightarrow$ rank differential matrix **R** $\left[\ \ \ \right]$

Ranking Vector

- Every ranking vector is a permutation
- Relative positions matter
- One-to-one mapping:

Example:

ranking vector $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ rank differential matrix **R**

Ranking Vector

- Every ranking vector is a permutation
- Differences in position have meaning
- One-to-one mapping:

Example:

ranking vector

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank differential matrix **R**

- Every rank differential matrix **R** is a reordering of the fundamental rank differential matrix $\hat{\mathbf{R}}$

$$\hat{\mathbf{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

corresponds to ranking vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Rank Differential Method

Input Matrix: **D** Output Matrix: **R**

Rank Differential Method

Input Matrix: $\mathbf{D}$ Output Matrix: $\mathbf{R}$

GOAL: Find symmetric reordering of $\mathbf{D}$ that

$$\min_{\mathbf{q}} \|\mathbf{D}(\mathbf{q},\mathbf{q}) - \hat{\mathbf{R}}\|$$

$\mathbf{D}$ = data differential matrix (e.g., Markov voting matrix $\mathbf{V}$)

$\mathbf{q}$ = permutation vector

$\hat{\mathbf{R}}$ = fundamental rank differential matrix

Rank Differential Example

$$\mathbf{D} = \begin{array}{c|ccccc} & \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \\ \text{Duke} & 0 & 0 & 0 & 0 & 0 \\ \text{Miami} & 45 & 0 & 18 & 8 & 20 \\ \text{UNC} & 3 & 0 & 0 & 2 & 0 \\ \text{UVA} & 31 & 0 & 0 & 0 & 0 \\ \text{VT} & 45 & 0 & 27 & 38 & 0 \end{array}$$

$$\hat{\mathbf{R}} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 1 & 2 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

- Find ordering of teams that brings **D** closest to $\hat{\mathbf{R}}$

Rank Differential Example

$$\mathbf{D} = \begin{array}{c} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 45 & 0 & 18 & 8 & 20 \\ 3 & 0 & 0 & 2 & 0 \\ 31 & 0 & 0 & 0 & 0 \\ 45 & 0 & 27 & 38 & 0 \end{pmatrix} \end{array}$$

$$\hat{\mathbf{R}} = \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Find ordering of teams that brings **D** closest to $\hat{\mathbf{R}}$

NORMALIZE

Rank Differential Example

$$\mathbf{D} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ .19 & 0 & .08 & .03 & .08 \\ .01 & 0 & 0 & .01 & 0 \\ .13 & 0 & 0 & 0 & 0 \\ .19 & 0 & .11 & .16 & 0 \end{pmatrix} \end{array} \quad \hat{\mathbf{R}} = \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{pmatrix} 0 & .05 & .10 & .15 & .20 \\ 0 & 0 & .05 & .10 & .15 \\ 0 & 0 & 0 & .05 & .10 \\ 0 & 0 & 0 & 0 & .05 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Find ordering of teams that brings **D** closest to $\hat{\mathbf{R}}$

NORMALIZE

Rank Differential Example

$\mathbf{D} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	.19	0	.08	.03	.08
UNC	.01	0	0	.01	0
UVA	.13	0	0	0	0
VT	.19	0	.11	.16	0

$\hat{\mathbf{R}} =$

	1	2	3	4	5
1	0	.05	.10	.15	.20
2	0	0	.05	.10	.15
3	0	0	0	.05	.10
4	0	0	0	0	.05
5	0	0	0	0	0

- Optimal ordering: $\mathbf{q} = [5 \quad 2 \quad 4 \quad 3 \quad 1]$

$\mathbf{D}(\mathbf{q}, \mathbf{q}) =$

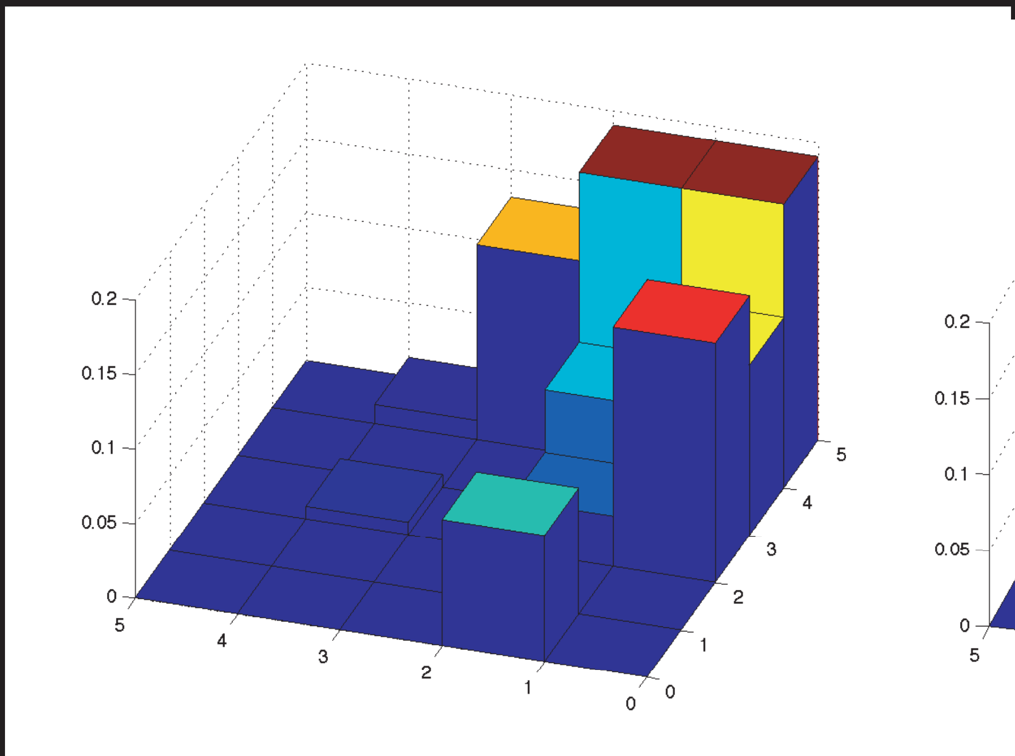
	Duke	Miami	UNC	UVA	VT
Duke	0	0	.16	.11	.19
Miami	.08	0	.03	.01	.16
UNC	0	0	0	0	.13
UVA	0	0	.01	0	.01
VT	0	0	0	0	0

$\hat{\mathbf{R}} =$

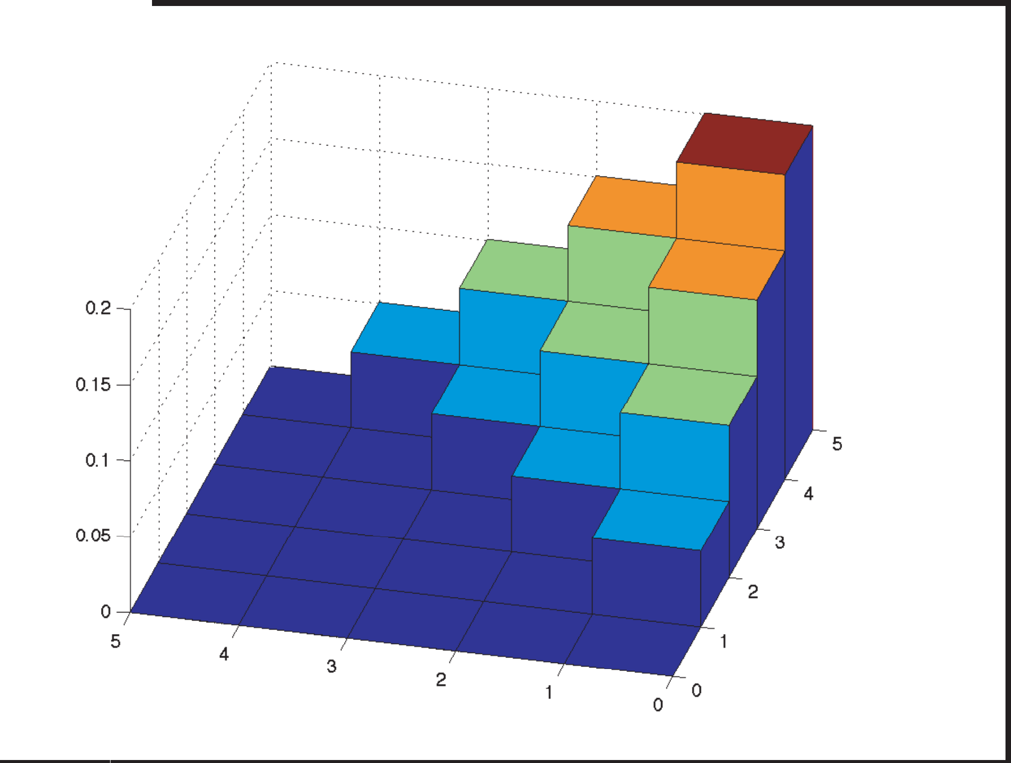
	1	2	3	4	5
1	0	.05	.10	.15	.20
2	0	0	.05	.10	.15
3	0	0	0	.05	.10
4	0	0	0	0	.05
5	0	0	0	0	0

Rank Differential Example

Reordered **D**

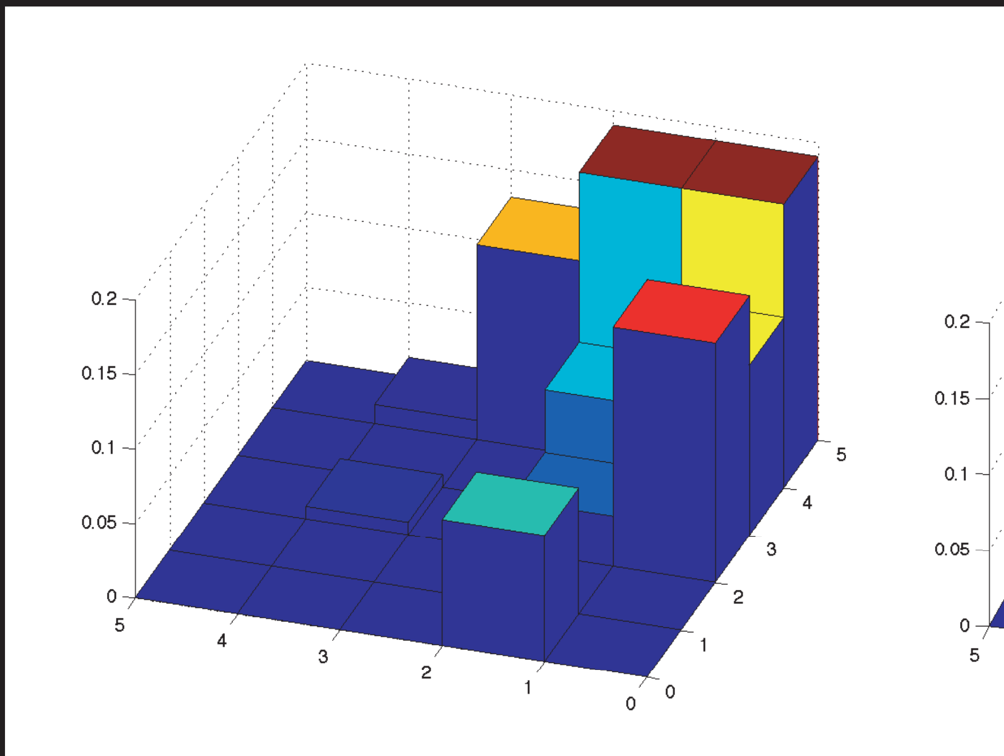


R̂

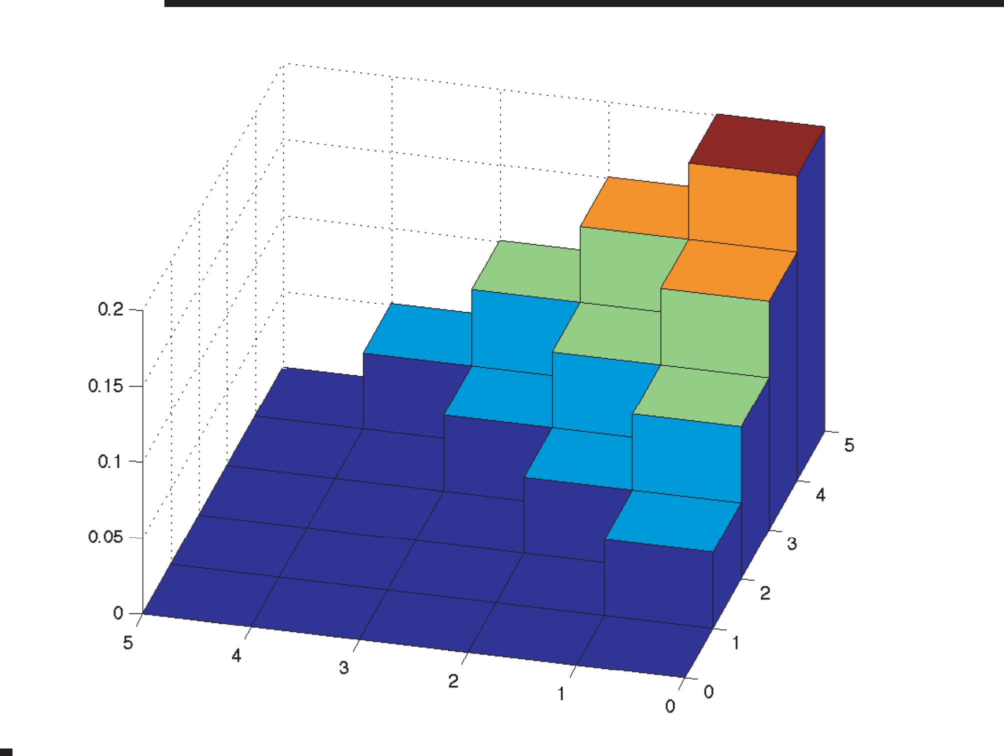


Rank Differential Example

Reordered **D**



$\hat{\mathbf{R}}$



rank differential method

Duke	5^{th}
Miami	2^{nd}
UNC	3^{rd}
UVA	4^{th}
VT	1^{st}

mHITS method

Duke	5^{th}
Miami	2^{nd}
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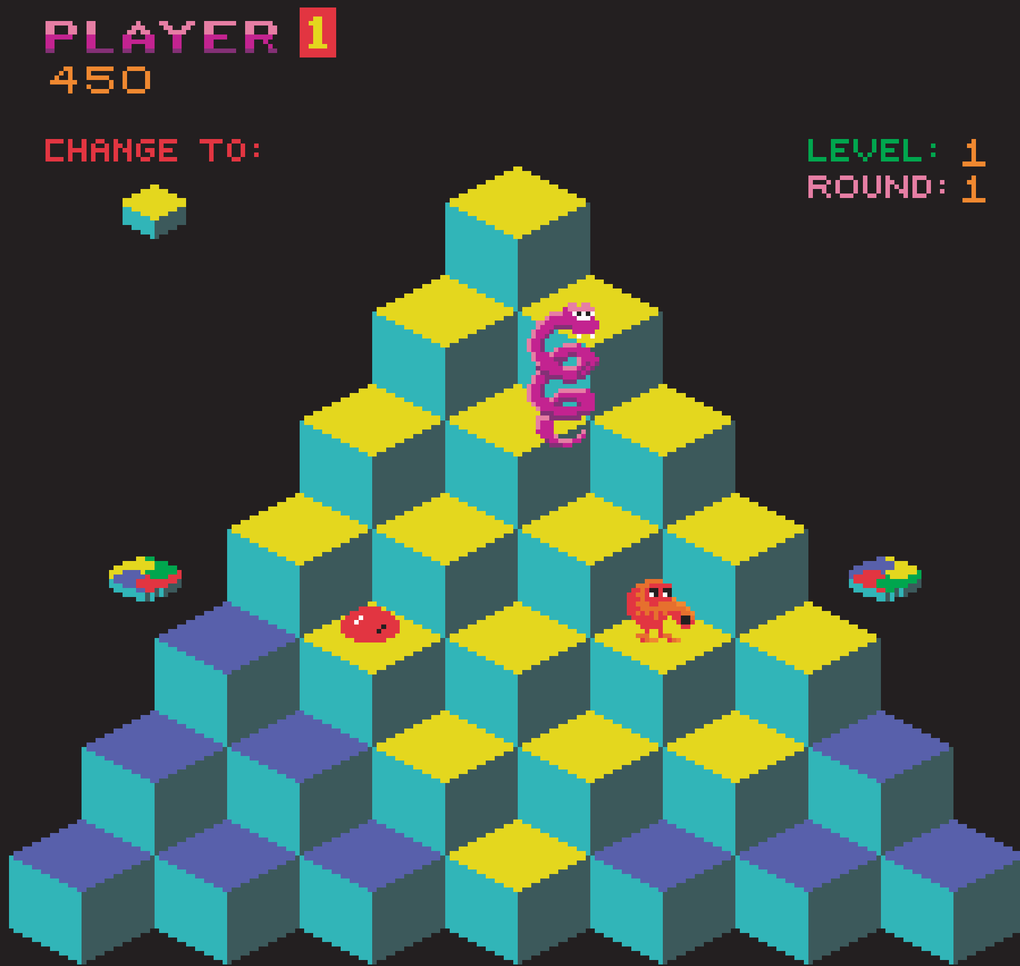
VT	1^{st}

Gottlieb 1982

Gottlieb 1982



Solving the Optimization Problem

$$\min_{\mathbf{q}} \ \|\mathbf{D}(\mathbf{q},\mathbf{q}) - \hat{\mathbf{R}}\|$$

- Huge solution space: $\exists\, n!$ permutations $\mathbf{q}$ (related to TSP; NP-hard)

EVOLUTIONARY ALGORITHM

initialize population with k=10 solutions $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k\}$
until convergence
  compute fitness $\|\mathbf{D}(\mathbf{x}_i,\mathbf{x}_i)-\hat{\mathbf{R}}\|$ for each $\mathbf{x}_i$
  create new population by
    copy 3 fittest $\mathbf{x}_i$ into next generation
    pair 6 fittest $\mathbf{x}_i$ and mate with rank aggregation (coming soon)
    mutate next 3 $\mathbf{x}_i$ with flip, invert, reversal operators
    insert 1 immigrant $\mathbf{x}_i$ using random permutation
end

guaranteed to converge to global min (Fogel, Michelawicz) but slow

New Ranking Methods

Rating Differential Method

Rating Differential Method

- Rank mapping:

ranking vector $\longleftrightarrow$ rank differential matrix **R**

$$\begin{bmatrix} \\ \end{bmatrix} \longleftrightarrow \begin{bmatrix} \\ \end{bmatrix}$$

- Rating Mapping:

rating vector $\longleftrightarrow$ rating differential matrix **R**

$$\begin{bmatrix} \\ \end{bmatrix} \longleftrightarrow \begin{bmatrix} \\ \end{bmatrix}$$

Rating Differential Method

- Rank mapping:

ranking vector $\begin{bmatrix} \\ \\ \\ \end{bmatrix} \longleftrightarrow$ rank differential matrix **R** $\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

- Rating Mapping:

Example:

rating vector $\begin{bmatrix} -4 \\ 7 \\ 1 \end{bmatrix} \longleftrightarrow$ rating differential matrix **R** $\begin{bmatrix} 0 & 0 & 0 \\ 11 & 0 & 6 \\ 5 & 0 & 0 \end{bmatrix}$

Rating Differential Method

- Rank mapping:

ranking vector $\longleftrightarrow$ rank differential matrix **R**

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} \longleftrightarrow \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

- Rating Mapping:

rating vector $\longleftrightarrow$ rating differential matrix **R**

Example:

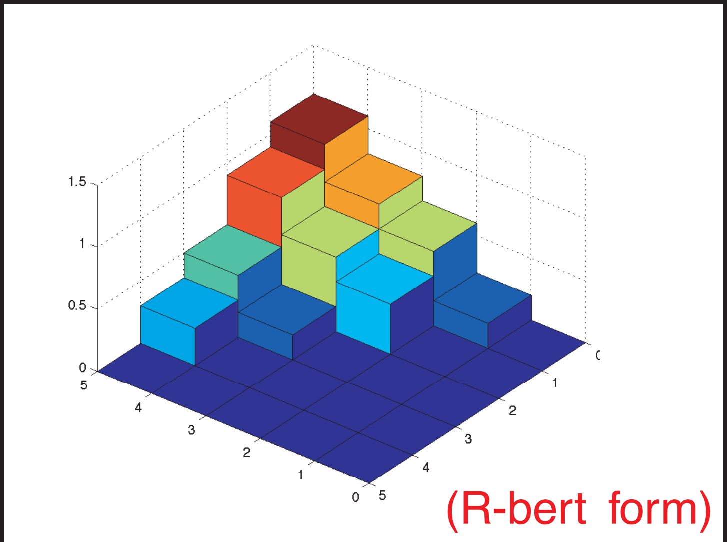
$$\begin{bmatrix} -4 \\ 7 \\ 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 11 & 0 & 6 \\ 5 & 0 & 0 \end{bmatrix}$$

- No fundamental rating differential matrix BUT there is a fundamental form for rating differential matrix

Rating Differential Fundamental Form

A rating differential matrix $\mathbf{R}$ is in *fundamental form* if

$$r_{ij} = 0\,, \quad \forall\, i \geq j \qquad \text{(strictly upper triangular)}$$
$$r_{ij} \leq r_{ik}\,, \quad \forall\, i \ni j \leq k \qquad \text{(ascending order across rows)}$$
$$r_{ij} \geq r_{kj}\,, \quad \forall\, j \ni i \leq k \qquad \text{(descending order down columns)}$$



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$$r_{ij} \geq r_{kj}\,, \quad \forall\, j \ni i \leq k \qquad \text{(descending order down columns)}$$

GOAL: Find symmetric reordering of **D** so that **D**(**q**,**q**)

$$\min_{\mathbf{q}} \; (\#\ \text{violations of fundamental form constraints})$$

Rating Differential Example

$$\mathbf{D} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 45 & 0 & 18 & 8 & 20 \\ 3 & 0 & 0 & 2 & 0 \\ 31 & 0 & 0 & 0 & 0 \\ 45 & 0 & 27 & 38 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{D}(\mathbf{q},\mathbf{q}) = \begin{array}{c} \\ \text{Miami} \\ \text{VT} \\ \text{UNC} \\ \text{UVA} \\ \text{Duke} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Miami} & \text{VT} & \text{UNC} & \text{UVA} & \text{Duke} \end{array} \\ \begin{pmatrix} 0 & 20 & 18 & 8 & 45 \\ 0 & 0 & 27 & 38 & 45 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Rating Differential Example

$$\mathbf{D} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 45 & 0 & 18 & 8 & 20 \\ 3 & 0 & 0 & 2 & 0 \\ 31 & 0 & 0 & 0 & 0 \\ 45 & 0 & 27 & 38 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{D}(\mathbf{q},\mathbf{q}) = \begin{array}{c} \\ \text{Miami} \\ \text{VT} \\ \text{UNC} \\ \text{UVA} \\ \text{Duke} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Miami} & \text{VT} & \text{UNC} & \text{UVA} & \text{Duke} \end{array} \\ \begin{pmatrix} 0 & 20 & 18 & 8 & 45 \\ 0 & 0 & 27 & 38 & 45 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 31 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

	mHITS method	rank differential method	rating differential method
Duke	5^{th}	5^{th}	5^{th}
Miami	2^{nd}	2^{nd}	1^{st}
UNC	4^{th}	3^{rd}	3^{rd}
UVA	3^{rd}	4^{th}	4^{th}
VT	1^{st}	1^{st}	2^{nd}

Solving the Optimization Problem

$$\min_{\mathbf{q}} \left(\#\ \text{violations of fundamental form constraints}\right)$$

- Huge solution space: $\exists\ n!$ permutations $\mathbf{q}$ (related to TSP; NP-hard)

EVOLUTIONARY ALGORITHM

---

initialize population with k=10 solutions $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k\}$

until convergence

compute fitness $\|\mathbf{D}(\mathbf{x}_i,\mathbf{x}_i)-\hat{\mathbf{R}}\|$ for each $\mathbf{x}_i$

create new population by

copy 3 fittest $\mathbf{x}_i$ into next generation

pair 6 fittest $\mathbf{x}_i$ and mate with rank aggregation

mutate next 3 $\mathbf{x}_i$ with flip, invert, reversal operators

insert 1 immigrant $\mathbf{x}_i$ using random permutation

end

---

guaranteed to converge to global min (Fogel, Michelawicz) but slow

Solving the Optimization Problem

$$\min_{\mathbf{q}} \ (\text{\# violations of fundamental form constraints})$$

- Huge solution space: $\exists\ n!$ permutations $\mathbf{q}$ (related to TSP; NP-hard)

EVOLUTIONARY ALGORITHM

initialize population with k=10 solutions $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k\}$
until convergence
    compute fitness # violations for each $\mathbf{x}_i$
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        copy 3 fittest $\mathbf{x}_i$ into next generation
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end

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Aggregation

Aggregation

Methods of Comparison

Methods of Comparison

Many methods, which is best?

Qualitative

- $\Re^1$ plots
- bipartite line graphs

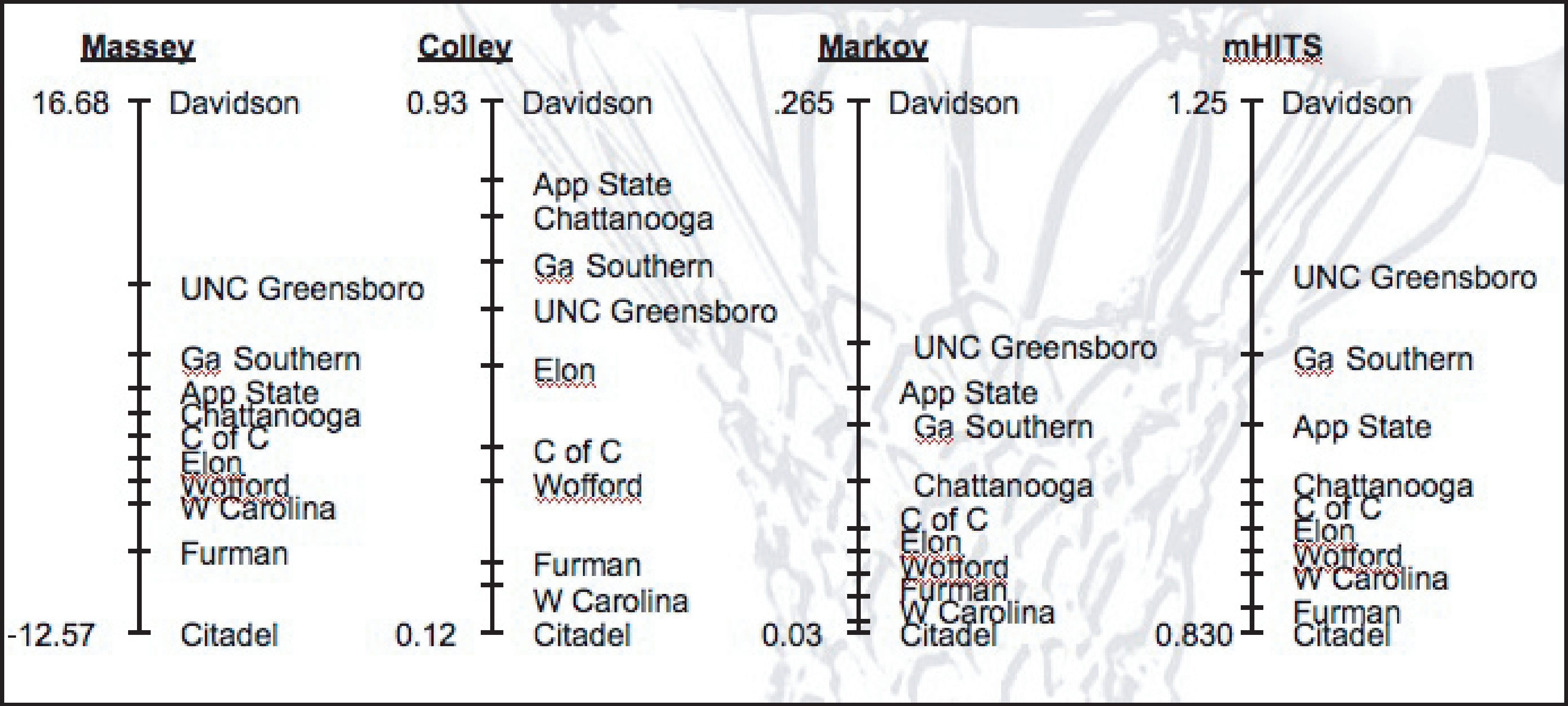
Quantitative

- distance between two ranked lists
  - Kendall's τ
  - Spearman's footrule
- distance to aggregated list

Methods of Comparison

$\Re^1$ plots

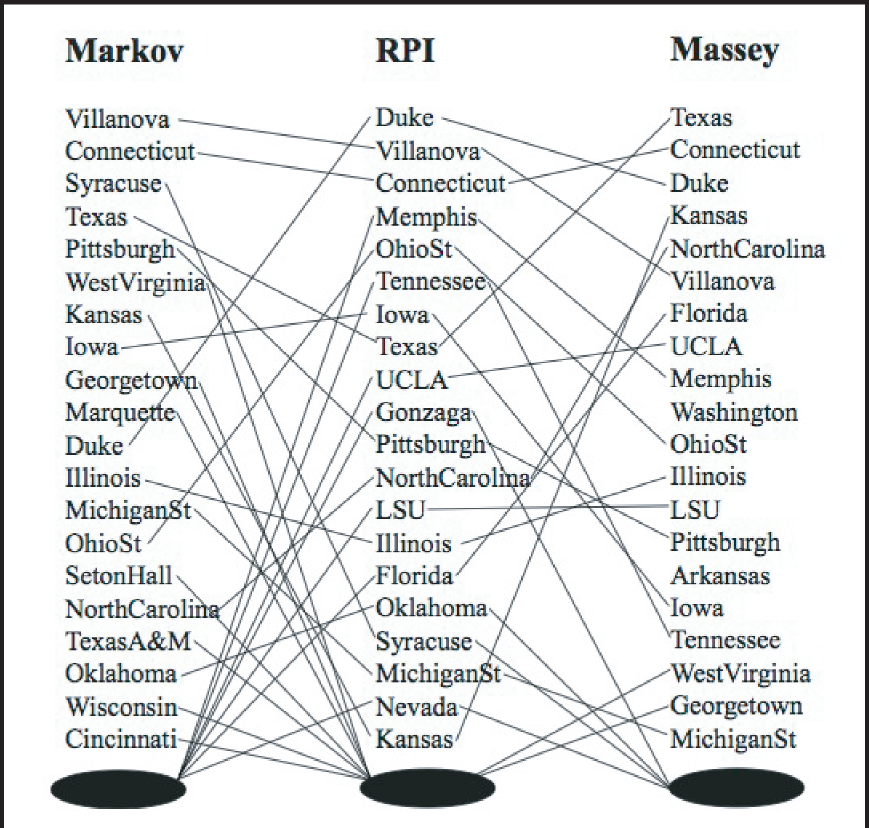
2008 SoCon results (Neil Goodson and Colin Stephenson)



Methods of Comparison

bipartite line graphs

2005 NCAA basketball



Methods of Comparison

distance between two ranked lists

Kendall's τ on full lists of length n

$$-1 \leq \tau = \frac{n_c - n_d}{\binom{n}{2}} \leq 1$$

n_c = # concordant pairs

n_d = # discordant pairs

$\tau = 1$, lists in complete agreement

$\tau = -1$, one list is reverse of the other

Methods of Comparison

distance between two ranked lists

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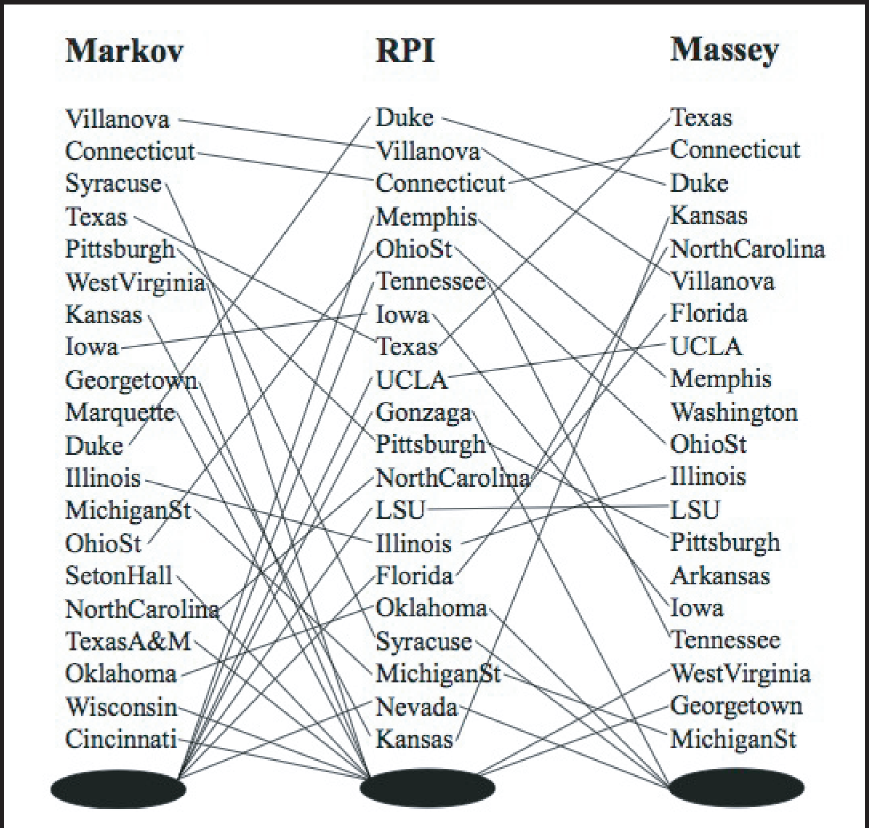
What about partial lists, like top-k lists?

Methods of Comparison

bipartite line graphs

2005 NCAA basketball



Kendall's Tau on partial lists

$$\tau = \frac{n_c - n_d - n_u}{\binom{n}{\mathbf{2}} - n_u}$$

n_c = # concordant pairs

n_d = # discordant pairs

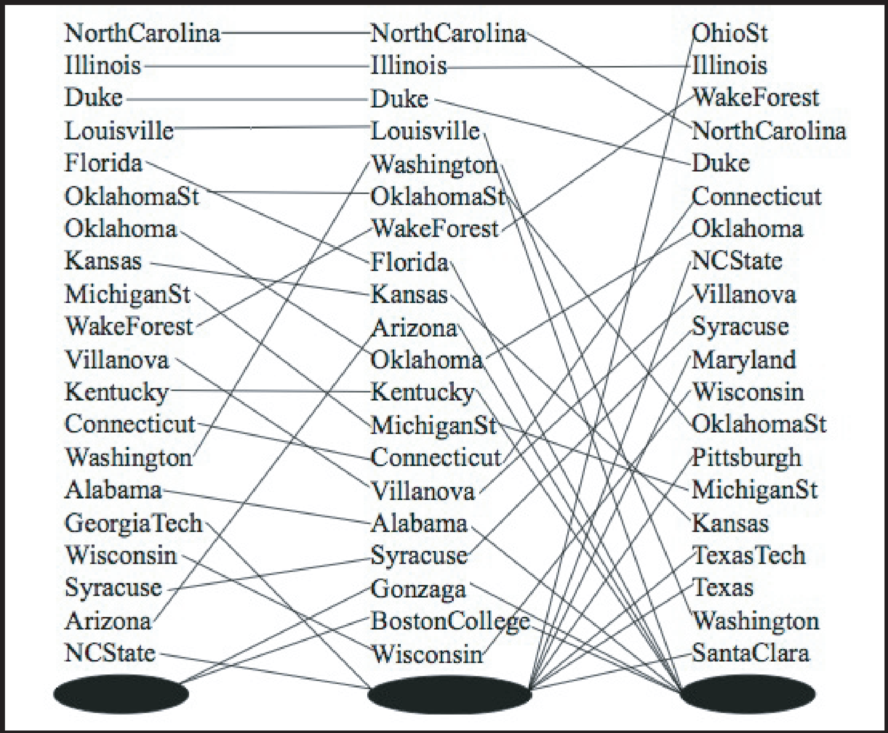
n_u = # unlabeled pairs

Bounds

$$\frac{-\binom{n}{\mathbf{2}}}{\binom{n}{\mathbf{2}} - n_u} \leq \tau \leq \frac{\binom{n}{\mathbf{2}}}{\binom{n}{\mathbf{2}} - n_u}$$

Kendall's Tau on partial lists

τ=.67 τ=.06



Methods of Comparison

distance between two ranked lists

Spearman's footrule on full lists l and $\tilde{l}$ of length n

$$\mathbf{0} \leq \phi = \sum_{i=1}^{n} |l(i) - \tilde{l}(i)|$$

$\phi = \|l - \tilde{l}\|_1$

Methods of Comparison

distance between two ranked lists

Spearman's footrule on full lists l and $\tilde{l}$ of length n

$$0 \leq \phi = \sum_{i=1}^{n} |l(i) - \tilde{l}(i)|$$

$\phi = \|l - \tilde{l}\|_1$

BUT disagreements in lists are given equal weight

Methods of Comparison

distance between two ranked lists

Weighted footrule on full lists l and $\tilde{l}$ of length n

$$\phi = \frac{\sum_{i=1}^{n} |l(i) - \tilde{l}(i)|}{min\{l(i), \tilde{l}(i)\}}$$

Methods of Comparison

distance between two ranked lists

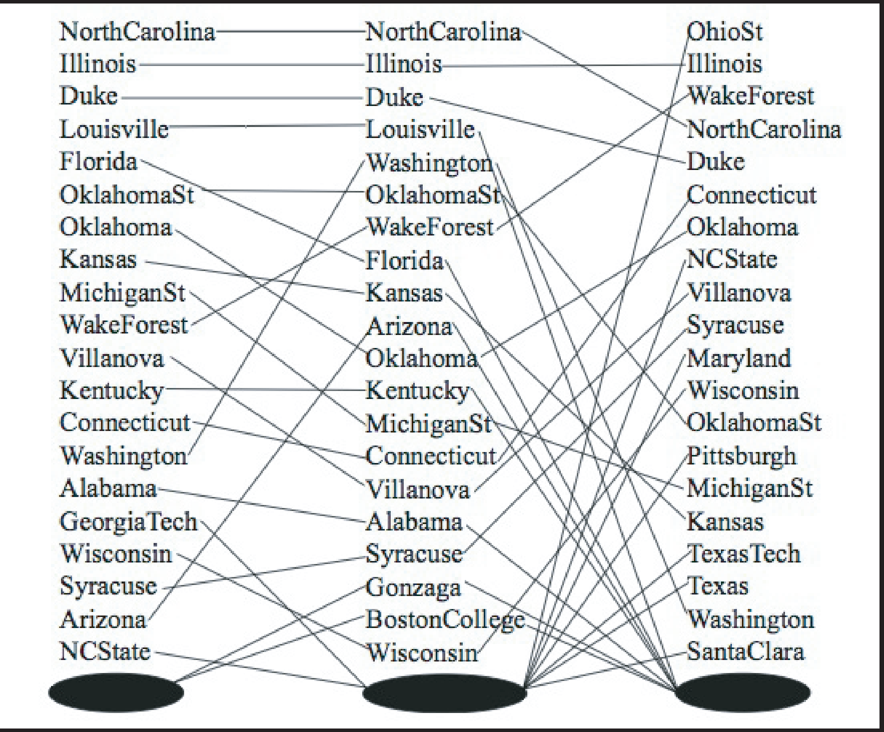
Weighted footrule on full lists l and $\tilde{l}$ of length n

$$\phi = \frac{\sum_{i=1}^{n} |l(i) - \tilde{l}(i)|}{min\{l(i), \tilde{l}(i)\}}$$

What about partial lists, like top-k lists?

Weighted Footrule on partial lists

ϕ=.05 ϕ=.27



Weighted Footrule on partial lists

Weighted Footrule ϕ Measure for Comparing Partial Lists of Length k

The weighted footrule measure ϕ between two *partial* lists l and $\tilde{l}$, both of length k, is built from individual ϕ_i values and normalized so that $0 \leq \phi \leq 1$.

$$\phi = \frac{\sum_{i=1}^{k} \phi_i}{\phi(l, l^c)},$$

where

$$\phi(l, l^c) = -2k + 2x \sum_{i=1}^{k} 1/i.$$

Each item i belongs to one of two following classes, and thus it's contribution ϕ_i to ϕ is calculated accordingly.

- For item $i \in l \cap \tilde{l}$ (i.e, an item appearing in both lists l and $\tilde{l}$),

$$\phi_i = \frac{|l(i) - \tilde{l}(i)|}{min(l(i), \tilde{l}(i))}.$$

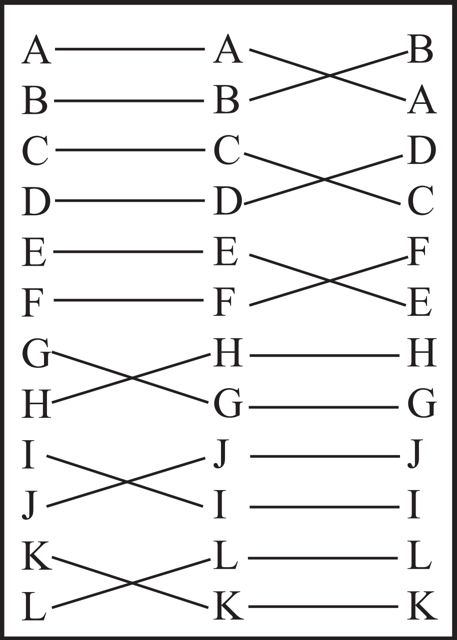
- For item $i \in (l \cup \tilde{l})/(l \cap \tilde{l})$ (i.e., an item appearing in only one list, which without loss of generality, we assume is l),

$$\phi_i = \frac{|l(i) - x|}{min(l(i), x)},$$

where x is defined as

$$x = \frac{k - 4\lfloor k/2 \rfloor + 2(k+1) \sum_{i=1}^{\lfloor k/2 \rfloor} 1/i}{\sum_{i=1}^{k} 1/i}.$$

Weighted Footrule vs. Kendall Tau



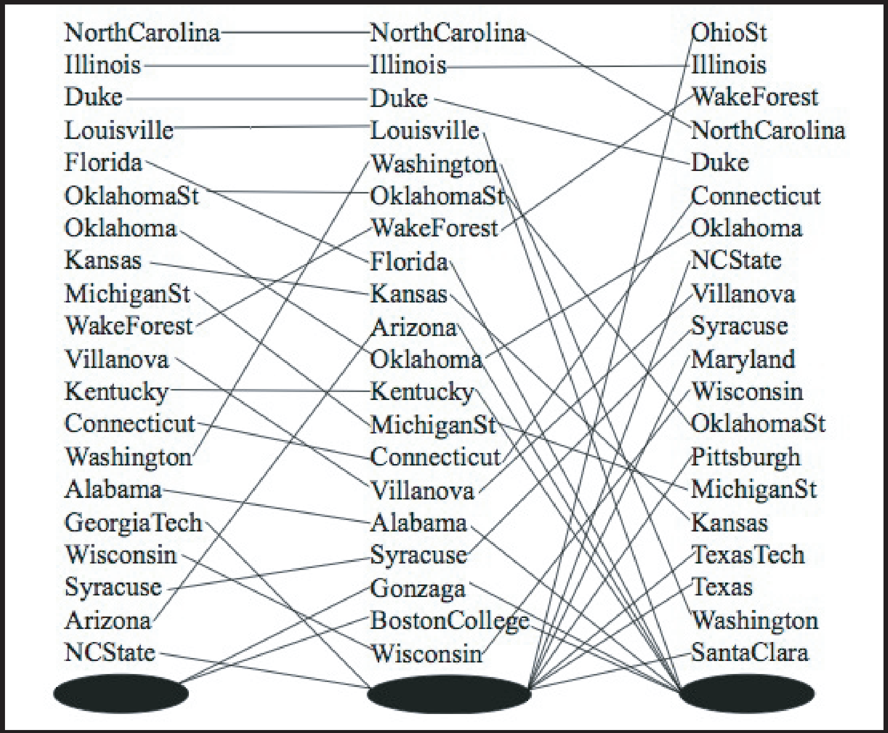
Kendall Tau: τ=.95 τ=.95

weighted footrule: ϕ=.34 ϕ=1.53

But which list is best? Methods of Comparison

distance to aggregated list

ϕ=.05 ϕ=.27

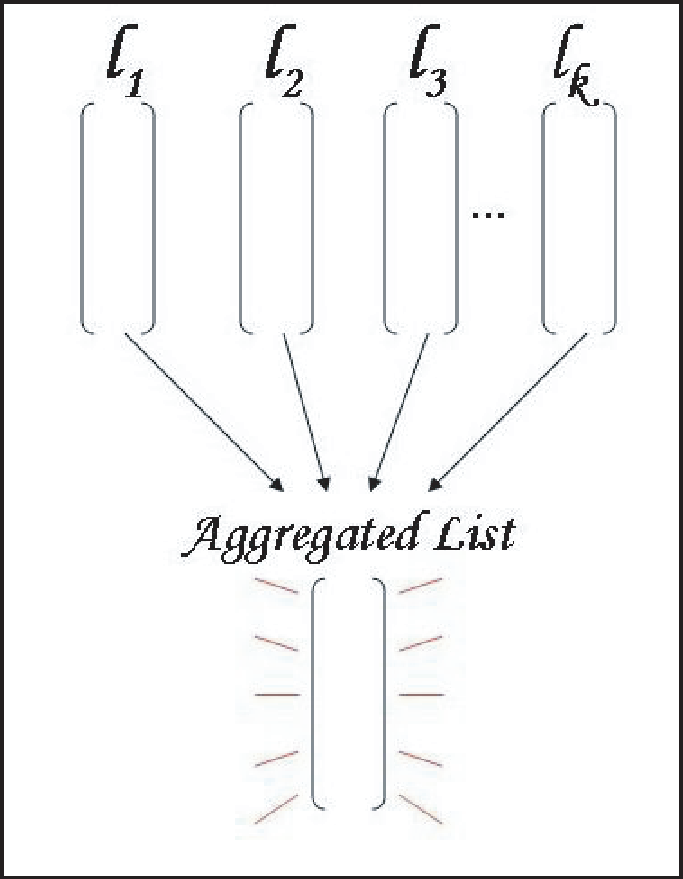


mHITS Aggregate Markov

Aggregation

Rank Aggregation

Rank Aggregation



Rank Aggregation

- average rank
- Borda count
- simulated data
- graph theory

Average Rank

	mHITS ($r = o/d$)	Massey	Colley	Average Rating	Average Rank
Duke	5th	5th	5th	5	5th
Miami	2nd	1st	1st	$1.\overline{3}$	1st
UNC	4th	4th	3rd	$3.\overline{6}$	4th
UVA	3rd	3rd	4th	$3.\overline{3}$	3rd
VT	1st	2nd	2nd	$1.\overline{6}$	2nd

Borda Count

- for each ranked list, each item receives a score equal to the number of items it outranks.

	mHITS ($r = o/d$)	Massey	Colley	Borda Count	Borda Rank
Duke	0	0	0	0	5th
Miami	3	4	4	11	1st
UNC	1	1	2	4	4th
UVA	2	2	1	5	3rd
VT	4	3	3	10	2nd

Simulated Data

3 ranked lists

	mHITS		*Massey*		*Colley*
1st	VT	1st	Miami	1st	Miami
2nd	Miami	2nd	VT	2nd	VT
3rd	UVA	3rd	UVA	3rd	UNC
4th	UNC	4th	UNC	4th	UVA
5th	Duke	5th	Duke	5th	Duke

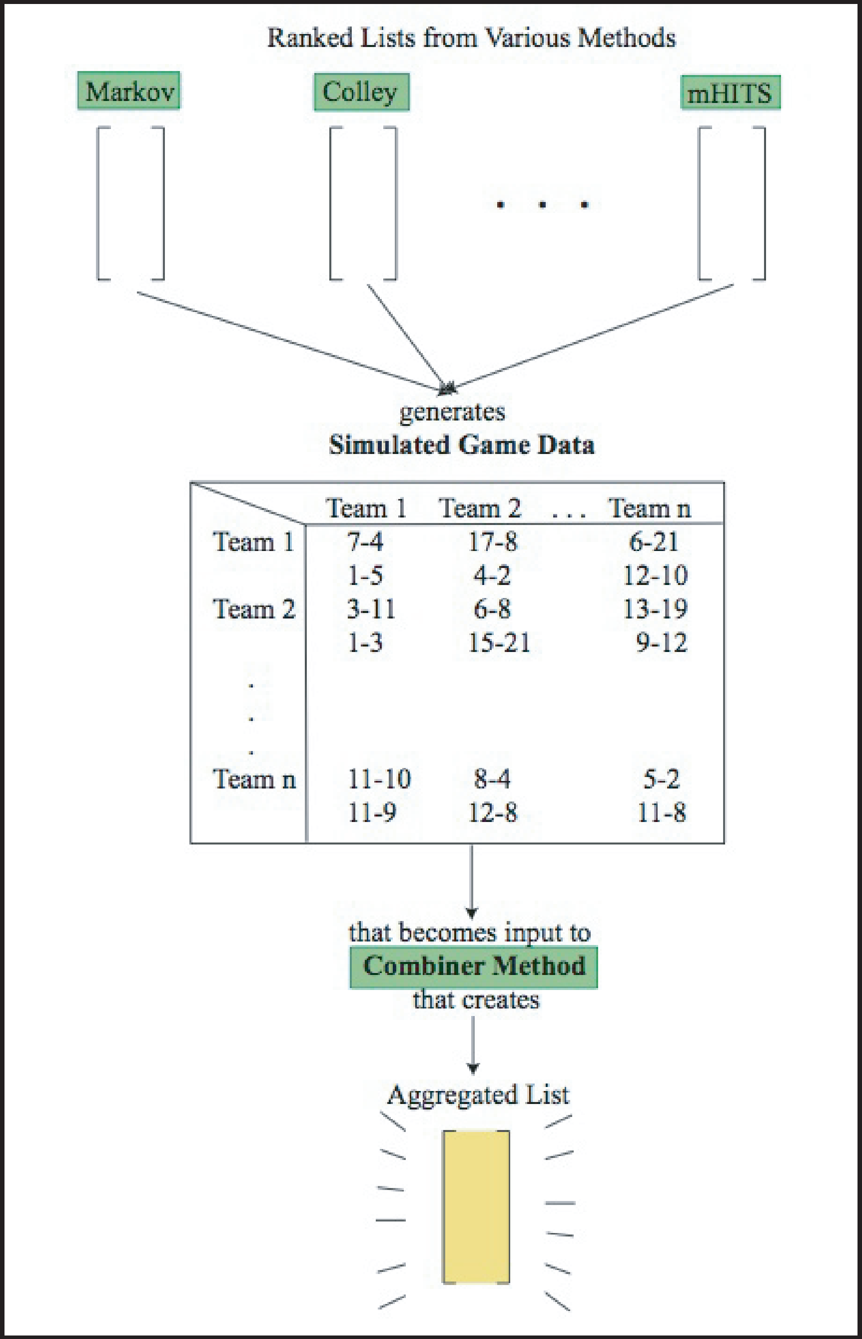
mHITS

- VT beats Miami by 1 point, UVA by 2 points, . . .
- Miami beats UVA by 1 point, UNC by 2 points, . . .
- UVA beats UNC by 1 point, Duke by 2 points
- UNC beats Duke by 1 point

repeat for each ranked list $\Rightarrow$ generates game scores for teams

Simulated Game Data

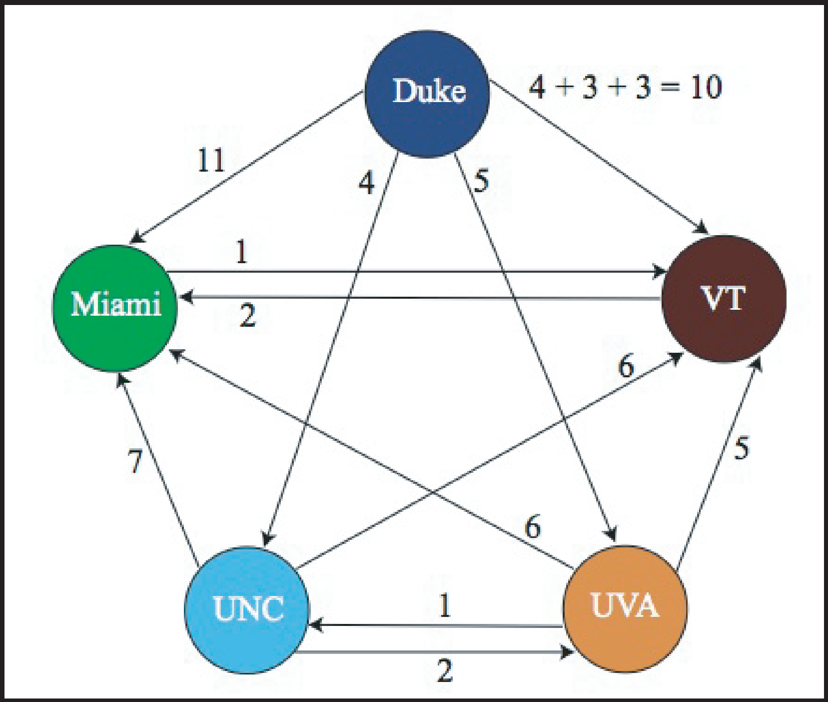
Simulated Data



Graph Theory

more voting

- ranked lists are used to form weighted graph
- possible weights
    - w_{ij} = # of ranked lists having i below j
    - w_{ij} = sum of rank differences of lists having i below j



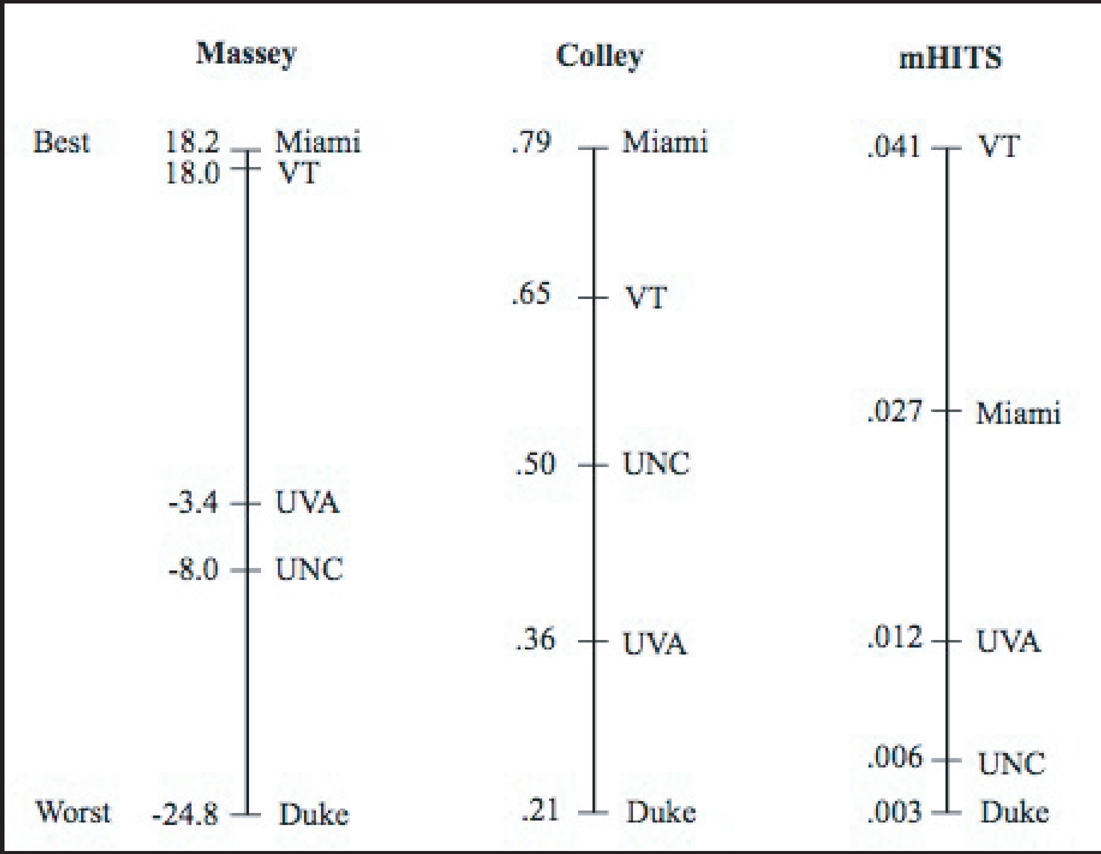
- run algorithm (e.g., Markov, PageRank, HITS) to determine most important nodes

Aggregation

Rating Aggregation

Rating Aggregation

rating vectors



- form rating differential matrix **R** for each rating vector

Rating Aggregation

rating differential matrices $\mathbf{R} \geq 0$

$$\mathbf{R}_{Massey} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 43 & 0 & 26.2 & 21.6 & .2 \\ 32.8 & 0 & 0 & 0 & 0 \\ 21.4 & 0 & 4.6 & 0 & 0 \\ 42.8 & 0 & 26 & 21.4 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{R}_{Colley} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ .58 & 0 & .29 & .43 & .14 \\ .29 & 0 & 0 & .14 & 0 \\ .15 & 0 & 0 & 0 & 0 \\ .44 & 0 & .15 & .29 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{R}_{mHITS} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ .024 & 0 & .021 & .015 & 0 \\ .003 & 0 & 0 & 0 & 0 \\ .009 & 0 & .006 & 0 & 0 \\ .038 & .014 & .035 & .029 & 0 \end{pmatrix} \end{array}$$

- differing scales $\longrightarrow$ NORMALIZE

Rating Aggregation

rating differential matrices

$\bar{\mathbf{R}}_{Massey} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	.1792	0	.1092	.09	.0008
UNC	.1367	0	0	0	0
UVA	.0892	0	.0192	0	0
VT	.1783	0	.1083	.0892	0

$\bar{\mathbf{R}}_{Colley} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	.2	0	.1	.1483	.0483
UNC	.1	0	0	.0483	0
UVA	.0517	0	0	0	0
VT	.1517	0	.0517	.1	0

$\bar{\mathbf{R}}_{mHITS} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	.1237	0	.1082	.0773	0
UNC	.0155	0	0	0	0
UVA	.0464	0	.0309	0	0
VT	.1959	.0722	.1804	.1495	0

Rating Aggregation

rating differential matrices

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$\bar{\mathbf{R}}_{Colley} =$

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Miami	.2	0	.1	.1483	.0483
UNC	.1	0	0	.0483	0
UVA	.0517	0	0	0	0
VT	.1517	0	.0517	.1	0

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	Duke	Miami	UNC	UVA	VT
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- combine into one matrix $\longrightarrow$ AVERAGE

Rating Aggregation

rating differential matrices

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$\bar{\mathrm{R}}_{Colley} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	.2	0	.1	.1483	.0483
UNC	.1	0	0	.0483	0
UVA	.0517	0	0	0	0
VT	.1517	0	.0517	.1	0

$\bar{\mathrm{R}}_{mHITS} =$

	Duke	Miami	UNC	UVA	VT
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UNC	.0155	0	0	0	0
UVA	.0464	0	.0309	0	0
VT	.1959	.0722	.1804	.1495	0

- combine into one matrix $\longrightarrow$ AVERAGE

$\bar{\mathrm{R}}_{average} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	0.1676	0	0.1058	0.1052	0.0164
UNC	0.0841	0	0	0.0161	0
UVA	0.0624	0	0.0167	0	0
VT	0.1753	0.0241	0.1135	0.1129	0

Rating Aggregation

average rating differential matrix $\mathbf{R}_{average} \geq 0$

$\bar{\mathbf{R}}_{average} =$

	Duke	Miami	UNC	UVA	VT
Duke	0	0	0	0	0
Miami	0.1676	0	0.1058	0.1052	0.0164
UNC	0.0841	0	0	0.0161	0
UVA	0.0624	0	0.0167	0	0
VT	0.1753	0.0241	0.1135	0.1129	0

- run ranking method
  - Markov method on $\mathbf{R}^T_{average}$
  - row sums of $\mathbf{R}_{average}$ / col sums of $\mathbf{R}_{average}$
  - Perron vector of $\mathbf{R}_{average}$

Rating Aggregation

average rating differential matrix $\mathbf{R}_{average} \geq 0$

$$\bar{\mathbf{R}}_{average} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.1676 & 0 & 0.1058 & 0.1052 & 0.0164 \\ 0.0841 & 0 & 0 & 0.0161 & 0 \\ 0.0624 & 0 & 0.0167 & 0 & 0 \\ 0.1753 & 0.0241 & 0.1135 & 0.1129 & 0 \end{pmatrix} \end{array}$$

- run ranking method
  - Markov method on $\mathbf{R}^T_{average}$
  - row sums of $\mathbf{R}_{average}$ / col sums of $\mathbf{R}_{average}$
  - Perron vector of $\mathbf{R}_{average}$

Team	Method 1 $\mathbf{r} = \mathbf{o}/\mathbf{d}$		Method 2 Markov $\mathbf{r}$		Method 3 Perron $\mathbf{r}$	
Duke	0	5^{th}	.020	5^{th}	.27	5^{th}
Miami	16.4	2^{nd}	.465	2^{nd}	.58	2^{nd}
UNC	.4	3^{rd}	.025	3^{rd}	.34	3^{rd}
UVA	.3	4^{th}	.024	4^{th}	.33	4^{th}
VT	26.0	1^{st}	.466	1^{st}	.61	1^{st}

Conclusions

- several methods for rating and ranking items begin by building nonnegative matrices
  - Markov: stationary vector of $\mathbf{V} \geq 0$
  - mHITS: Sinkhorn-Knopp on $\mathbf{P} \geq 0$
  - Rank Differential: reordering of $\mathbf{D} \geq 0$
  - Rating Differential: reordering of $\mathbf{D} \geq 0$

- nonnegative matrix theory many times insures existence, uniqueness, convergence.

- sometimes nonnegativity is forced to guarantee these properties

- rank and rating aggregation also build nonnegative matrices: $\mathbf{W} \geq 0$, $\mathbf{R} \geq 0$